I.1.7 All ducks are the same color

Find the flaw in the "proof" of the following

proposition: All ducks are the same color.

proof: $n = 1$: There is only one duck, so there is only one color.

$n = m$: The set of ducks is one-to-one correspondent to $\{1, 2, \dots, m\}$, and we assume that all m ducks are the same color.

$n = m + 1$: Now we have $\{1, 2, \dots, m, m + 1\}$. Consider the subsets $\{1, 2, \dots, m\}$ and $\{2, \dots, m, m+1\}$. Each of these represent sets of m ducks, which are all the same color by the induction assumption. But this means that ducks #2 through m are all the same color, and ducks #1 and $m+1$ are the same color as, e.g., duck #2, and hence all ducks are the same color.

remark: This demonstration of the pitfalls of inductive reasoning is due to George Pólya (1888 - 1985), who used horses instead of ducks.

(2 points)

Solution

The problem lies with $n = 2$.

The induction step from $n = m$ to $n = m + 1$ relies on the fact that the subsets $\{1, 2, \dots, m\}$ and $\{2, 3, \dots, m + 1\}$ have common elements. But for $n = 2$ we have $m = 1$, and the the two sets are $\{1\}$ and $\{2\}$, which have no common elements!

$\Rightarrow$ In order for the proof to be valid, one first has to prove that any **two** ducks have the same color, which is not possible. 2pts