I.2.1. Pauli group

The Pauli matrices are complex 2×2 matrices defined as

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad ,$$

Now consider the set P_1 that consists of the Pauli matrices and their products with the factors -1 and $\pm i$:

$$P_1 = \{\pm\sigma_0, \pm i\sigma_0, \pm\sigma_1, \pm i\sigma_1, \pm\sigma_2, \pm i\sigma_2, \pm\sigma_3, \pm i\sigma_3\}$$

Show that this set of 16 elements forms a (nonabelian) group under matrix multiplication called the Pauli group. It plays an important role in quantum information theory.

(3 points)

Solution

The Pauli matrices obey

	σ_0	σ_1	σ_2	σ_3
σ_0	σ_0	σ_1	σ_2	σ_3
σ_1	σ_1	σ_0	iσ_3	-iσ_2
σ_2	σ_2	-iσ_3	σ_0	iσ_1
σ_3	σ_3	iσ_2	-iσ_1	σ_0

i.e., $\sigma_i\sigma_j$ equals either some σ_k or some σ_k times $\pm i$. 1pt

Now consider P_1:

	σ_0	$-\sigma_0$	$i\sigma_0$	$-i\sigma_0$	σ_1	$-\sigma_1$	$i\sigma_1$	$-i\sigma_1$	σ_2	$-\sigma_2$	...
σ_0	σ_0	$-\sigma_0$	$i\sigma_0$	$-i\sigma_0$	σ_1	$-\sigma_1$	$i\sigma_1$	$-i\sigma_1$	σ_2	$-\sigma_2$	...
$-\sigma_0$	$-\sigma_0$	σ_0	$-i\sigma_0$	$i\sigma_0$	$-\sigma_1$	σ_1	$-i\sigma_1$	$i\sigma_1$	$-\sigma_2$	σ_2	...
$i\sigma_0$	$i\sigma_0$	$-i\sigma_0$	$-\sigma_0$	σ_0	$i\sigma_1$	$-i\sigma_1$	$-\sigma_1$	σ_1	$i\sigma_2$	$-i\sigma_2$	...
$-i\sigma_0$	$-i\sigma_0$	$i\sigma_0$	σ_0	$-\sigma_0$	$-i\sigma_1$	$i\sigma_1$	σ_1	$-\sigma_1$	$-i\sigma_2$	$i\sigma_2$	...
σ_1	σ_1	$-\sigma_1$	$i\sigma_1$	$-i\sigma_1$	σ_0	...	...	...	...	...	...
...											

etc. Even without completing the table, we see that

(i) The set is closed under matrix multiplication, since $\sigma_i\sigma_j$ is always some σ_k times either 1 or $\pm i$.

(ii) Matrix multiplication is associative. 1pt

(iii) σ_0 is the unit element.

(iv) Each element has an inverse:

$$\begin{array}{rclrclrclrcl}
\sigma_0\sigma_0 & = & \sigma_0 & \sigma_1\sigma_1 & = & \sigma_0 & \sigma_2\sigma_2 & = & \sigma_0 & \sigma_4\sigma_4 & = & \sigma_0 \\
(-\sigma_0)(-\sigma_0) & = & \sigma_0 & (-\sigma_1)(-\sigma_1) & = & \sigma_0 & (-\sigma_3)(-\sigma_3) & = & \sigma_0 & (-\sigma_4)(-\sigma_4) & = & \sigma_0 \\
(i\sigma_0)(-i\sigma_0) & = & \sigma_0 & (i\sigma_1)(-i\sigma_1) & = & \sigma_0 & (i\sigma_3)(-i\sigma_3) & = & \sigma_0 & (i\sigma_4)(-i\sigma_4) & = & \sigma_0 \\
(-i\sigma_0)(i\sigma_0) & = & \sigma_0 & (-i\sigma_1)(i\sigma_1) & = & \sigma_0 & (-i\sigma_3)(i\sigma_3) & = & \sigma_0 & (-i\sigma_4)(i\sigma_4) & = & \sigma_0
\end{array}$$

1pt