**I.4.2. The space of rank-2 tensors**

a) Prove the theorem of ch.1 §4.3: Let $V$ be a vector space $V$ of dimension $n$ over $K$. Then the space of rank-2 tensors, defined via bilinear forms $f : V \times V \to K$, forms a vector space of dimension $n^2$.

b) Consider the space of bilinear forms $f$ on $V$ that is equivalent to the space of rank-2 tensors, and construct a basis of that space.

*hint:* On the space of tensors, define a suitable addition and multiplication with scalars, and construct a basis of the resulting vector space.

(5 points)

**Solution**

a) We know that the rank-2 tensors are one-to-one correspondent to bilinear forms $f(x, y)$. On the set of bilinear forms, define an addition by

$$(f + g)(x, y) := f(x, y) + g(x, y)$$

This makes the set of forms an additive group. Also define a multiplication with scalars $\lambda \in K$ by

$$(\lambda\, f)(x, y) := \lambda\, f(x, y)$$

This makes the set of forms a $K$-vector space. On the space of rank-2 tensors $t, u, \ldots$ this corresponds to defining the tensor $t + u$ as the tensor with coordinates

$$(t + u)_{ij} = t_{ij} + u_{ij} \qquad \text{1pt}$$

and the tensor $\lambda\, t$ as the one with coordinates

$$(\lambda\, t)_{ij} = \lambda\, t_{ij}$$

The space of tensors is now a $K$-vector space. 1pt

Consider the cartesian basis of $V$, i.e., a set of $n$ vectors $e_i$ with contravariant coordinates $(e_i)^k = \delta_i{}^k$. Analogously, construct $n^2$ tensors $E_{ij}$ with contravariant coordinates

$$(E_{ij})^{kl} = \delta_i{}^k\, \delta_j{}^l$$

Define a tensor $t$ as a linear combination of the $E_{ij}$,

$$t = \sum_{ij} \tau^{ij}\, E_{ij}$$

with coefficients $\tau^{ij} \in K$. This tensor has coordinates

$$t^{kl} = \sum_{ij} \tau^{ij}\, (E_{ij})^{kl} = \tau^{kl}$$

$\Rightarrow$ Any rank-2 tensor can be written as a linear combination of the $E_{ij}$, with the coordinates $t^{ij}$ of $t$ as the coefficients:

$$t = \sum_{ij} t^{ij}\, E_{ij}$$

$\Rightarrow$ The $E_{ij}$ span the space. 1pt

Now, in order for $t$ to be the null tensor, all of its coordinates must be zero, so $t = 0$ implies $t^{ij} = 0 \; \forall i, j$.

$\Rightarrow$ The $E_{ij}$ are linearly independent.

$\Rightarrow$ The $n^2$ rank-2 tensors $E_{ij}$ form a basis of the space of rank-2 tensors, and hence the space has dimension $n^2$.

1pt

b) Let $f_{ij}$ be the bilinear form that corresponds to the tensor $E_{ij}$ and evaluate it on a pair of co-basis vectors. Then

$$f_{ij}(e^k, e^l) = (E_{ij})^{kl} = \delta_i{}^k \, \delta_j{}^l$$

For arbitrary $x, y \in V$ we have

$$f_{ij}(x, y) = x_k y_l f_{ij}(e^k, e^l) = x_k y_l \, \delta_i{}^k \delta_j{}^l = x_i y_j$$

$\Rightarrow$ The set of $n^2$ bilinear forms $f_{ij}$ defined by

$$f_{ij}(x, y) = x_i y_j$$

forms a basis of the space of bilinear forms. 1pt