I.4.5. $\mathbb{R}$ as a metric space

Consider the reals $\mathbb{R}$ with $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $\rho(x, y) = |x - y|$. Show that this definition makes $\mathbb{R}$ a metric space.

(3 points)

Solution

Positive definiteness and symmetry are obvious. 1pt

Now prove the triangle inequality:

Proof. By definition of $|x|$ we have $xy \le |x| \cdot |y|\ \forall\, x, y \in \mathbb{R}$. Therefore,

$$0 \le 2(x-y)(z-y) + 2\,|x-y| \cdot |z-y| \qquad (*)$$

1pt

And hence

$$\begin{aligned}
(x-z)^2 &= x^2 - 2xz + z^2 \\
&\le x^2 - 2xz + z^2 + \underbrace{2(x-y)(z-y) + 2|x-y| \cdot |z-y|}_{\ge 0 \text{ by } (*)} \\
&= x^2 - 2xz + z^2 + 2(x-y)z - 2(x-y)y + 2|x-y| \cdot |z-y| \\
&= x^2 \cancel{-2xz} + z^2 \cancel{+2xz} - 2xy + 2y^2 - 2yz + 2|x-y| \cdot |z-y| \\
&= x^2 - 2xy + y^2 + y^2 - 2yz + z^2 + 2|x-y| \cdot |z-y| \\
&= (x-y)^2 + (y-z)^2 + 2|x-y| \cdot |z-y| \\
&= (|x-y| + |y-z|)^2
\end{aligned}$$

But $(x-z)^2 \ge 0$, and hence we have the triangle inequality

$$|x-z| \le |x-y| + |y-z|$$

$\square$

1pt