

I.4.6. Limits of sequences

- a) Show that a sequence in a metric space has at most one limit.

hint: Assume there are two limits, and use the triangle inequality to show that they must be the same.

- b) Show that every sequence with a limit is a Cauchy sequence.

(3 points)

Solution

- a) Prove uniqueness:

Proof. Let x_n be a sequence. Suppose $x_n \Rightarrow x^*$ and $x_n \Rightarrow y^*$. Then

$$\rho(x^*, y^*) \leq \rho(x^*, x_n) + \rho(y^*, x_n) \quad \forall x_n \quad \text{by the triangle equation}$$

But

$$\lim_{n \rightarrow \infty} \rho(x^*, x_n) = \lim_{n \rightarrow \infty} \rho(y^*, x_n) = 0$$

and hence

$$\rho(x^*, y^*) = 0 \quad \Rightarrow \quad x^* = y^*$$

□

1pt

- b) Prove Cauchy-ness:

Proof. Let x_n have a limit x^* : $x_n \Rightarrow x^*$. Then

$$\rho(x_n, x_m) \leq \rho(x_n, x^*) + \rho(x_m, x^*) \quad 1\text{pt}$$

Let $\delta > 0$. Then $\exists N \in \mathbb{N}$ such that $\rho(x_n, x^*) < \delta \quad \forall n > N$.

Now let $\epsilon > 0$ and choose $\delta = \epsilon/2$. Then $\exists N > 0$ such that

$$\rho(x_n, x_m) \leq \rho(x_n, x^*) + \rho(x_m, x^*) < \epsilon/2 + \epsilon/2 = \epsilon$$

provided $n, m > N$.

□

1pt