

**II.2.2. Another causal function**

The function considered in Problem II.2.1 is an example of a class of complex functions called *causal functions* that are important in the theory of many-particle systems. Another member of this class is

$$G(z) = i \left( \sqrt{z^2 - 1} - z \right)$$

Determine the spectral density and the reactive part of  $G(z)$ , and plot them for  $-3 < \omega < 3$ .

(3 points)

**Solution**

*Note:* The function  $f(z)$  in the handwritten solution is  $-i G(z)$  in the typeset problem (to be fixed when the solution gets typeset).

II.2.2)  $f(z) = \sqrt{z^2 - 1} - z = \sqrt{z+1} \sqrt{z-1} - z$

$$\begin{aligned} \underline{f''(u)} &= \frac{1}{i} \left[ \sqrt{u+1+i0} \sqrt{u-1+i0} - \sqrt{u+1-i0} \sqrt{u-1-i0} \right] - \frac{1}{i} (u+i0 - u-i0) \\ &= \frac{1}{i} \left[ \left( \theta(u>1) (u+1)^{1/2} + \theta(u<1) |u+1|^{1/2} i \right) \left( \theta(u>1) (u-1)^{1/2} + \theta(u<1) |u-1|^{1/2} i \right) - cc \right] \\ &= \theta(u>1) \theta(u<1) |u^2-1|^{1/2} + \underbrace{\theta(u<1) \theta(u>1)}_{=0} |u^2-1|^{1/2} \\ &= \underline{\underline{\theta(u^2 < 1) (1-u^2)^{1/2}}} \end{aligned}$$

$$\begin{aligned} \underline{f'(u)} &= -\frac{1}{i} [f(u+i0) - f(u-i0)] = i \operatorname{Re} f(u+i0) \\ &= \theta(u>1) \theta(u>1) \sqrt{u^2-1} - \theta(u<1) \theta(u<1) \sqrt{u^2-1} - u \\ &= \underline{\underline{\theta(u^2 < 1) (-u) + \theta(u^2 > 1) (\sqrt{u^2-1} - u)}} \end{aligned}$$

