0.2.4. **Functional derivative**

Let $F[\varphi]$ be a functional of a real-valued function $\varphi(x)$. For simplicity, let $x \in \mathbb{R}$; the generalization to more than one dimension is straightforward. We can (sloppily) define the *functional derivative* of F as

$$\frac{\delta F}{\delta \varphi(x)} := \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Big(F[\varphi(y) + \epsilon \delta(y-x)] - F[\varphi(y)] \Big)$$

a) Calculate $\delta F / \delta \varphi(x)$ for the following functionals:

   i) $F = \int dx \, \varphi(x)$

   ii) $F = \int dx \, \varphi^2(x)$

   iii) $F = \int dx \, f(\varphi(x)) \, g(\varphi(x))$ where f and g are given functions

   iv) $F = \int dx \, (\varphi'(x))^2$ where $\varphi'(x) = d\varphi/dx$
   hint: Integrate by parts and assume that the boundary terms vanish.

   v) $F = \int dx \, V(\varphi'(x))$ where V is some given function.

   remark: Blindly ignore terms that formally vanish as $\epsilon \to 0$ unless you want to find out why the above definition is very problematic. It does work for operational purposes, though.

b) Consider a Lagrangian density' $\mathcal{L}(\varphi(x), \partial_\mu \varphi(x))$ and an action' $S = \int d^4x \, \mathcal{L}$. Show that extremizing S by requiring $\delta S / \delta \varphi(x) \equiv 0$ with the above definition of the functional derivative leads to the Euler-Lagrange equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = \frac{\partial \mathcal{L}}{\partial \varphi}$$

(3 points)

Solution

a) i) $\frac{\delta F}{\delta \varphi(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dy \, [\varphi(y) + \epsilon \delta(y-x) - \varphi(y)] = \int dy \, \delta(y-x) = 1$

   ii) $\frac{\delta F}{\delta \varphi(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dy \left[(\varphi(y) + \epsilon \delta(y-x))^2 - \varphi(y)^2 \right] = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dy \, \left[2\epsilon \varphi(y) \delta(y-x) + O(\epsilon^2) \right]$
   $= 2\varphi(x)$

   iii) $\frac{\delta F}{\delta \varphi(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dy \, [f(\varphi(y) + \epsilon \delta(y-x))] \, [g(\varphi(y) + \epsilon \delta(y-x)) - f(\varphi(y)) \, g(\varphi(y))]$
   $= f'(\varphi(x)) g(\varphi(x)) + f(\varphi(x)) g'(\varphi(x))$ 1pt

   iv) $\frac{\delta F}{\delta \varphi(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dy \left[\left(\varphi'(y) + \epsilon \frac{d}{dy} \delta(y-x) \right)^2 - (\varphi'(y))^2 \right] = 2 \int dy \, \varphi'(y) \frac{d}{dy} \delta(y-x) = -2\varphi''(x)$

   v) $\frac{\delta F}{\delta \varphi(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dy \left[V \left(\varphi'(y) + \epsilon \frac{d}{dy} \delta(y-x) \right) - V(\varphi'(y)) \right]$
   $= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dy \left[\epsilon V'(\varphi'(y)) \frac{d}{dy} \delta(y-x) + O(\epsilon^2) \right] = -V''(\varphi'(x)) \, \varphi''(x)$ 1pt

b) $0 = \frac{\delta}{\delta \varphi(x)} \int d^4y \, \mathcal{L}(\varphi(y), \partial_\mu \varphi(y))$
   $= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int d^4y \, [\mathcal{L}(\varphi(y) + \epsilon \delta(y-x), \partial_\mu \varphi(y) + \epsilon \partial_\mu \delta(y-x)) - \mathcal{L}(\varphi(y), \partial_\mu \varphi(y))]$
   $= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int d^4y \left[\epsilon \delta(y-x) \frac{\partial \mathcal{L}}{\partial \varphi(y)} + \epsilon \left(\partial_\mu \delta(y-x) \right) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi(y))} + O(\epsilon^2) \right]$
   $= \frac{\partial \mathcal{L}}{\partial \varphi(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi(x))}$ EL ✓ 1pt