0.3.1. **Particle in homogeneous $\boldsymbol{E}$ and $\boldsymbol{B}$ fields**

Consider a point particle (mass m, charge e) in homogeneous fields $\boldsymbol{B} = (0, 0, B)$ and $\boldsymbol{E} = (0, E_y, E_z)$. Treat the motion of the particle nonrelativistically.

a) Show that the motion in z-direction decouples from the motion in the x-y plane, and find $z(t)$.

b) Consider $\xi := x + iy$. Find the equation of motion for ξ, and its most general solution.

   hint: Define the *cyclotron frequency* $\omega = eB/mc$, and remember how to solve inhomogeneous ODEs.

c) Show that the time-averaged velocity perpendicular to the plane defined by $\boldsymbol{B}$ and $\boldsymbol{E}$ is given by the *drift velocity*

$$\langle \boldsymbol{v} \rangle = c\, \boldsymbol{E} \times \boldsymbol{B}/\boldsymbol{B}^2$$

   Show that $E_y/B \ll 1$ is necessary and sufficient for the non relativistic approximation to be valid.

d) Show that the path projected onto the x-y plane can have three qualitatively different shapes, and plot a representative example for each.

(6 points)

0.3.1.) a) Eq. of motion: $m\dot{\vec{v}} = e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B}$

let $\vec{B} = (0,0,B)$ and $\vec{E} = (0, E_y, E_z)$

$$\rightarrow \boxed{\begin{aligned} m\ddot{x} &= \frac{e}{c}\dot{y}B && (1)\\ m\ddot{y} &= eE_y - \frac{e}{c}\dot{x}B && (2)\\ m\ddot{z} &= eE_z && (3)\end{aligned}}$$

(1) $(3) \rightarrow \underline{\underline{z(t) = z_0 + v_z^0 t + \frac{eE_z}{2m}t^2}}$

b) Define $\underline{\xi := x + iy}$

$(1) + i\cdot(2) \rightarrow m\ddot{\xi} = ieE_y - i\frac{eB}{c}\dot{\xi}$

Define $\omega := \frac{eB}{mc}$ cyclotron frequency

$$\rightarrow \boxed{\ddot{\xi} + i\omega\dot{\xi} = i\frac{e}{m}E_y} \quad (*)$$

Special solution of inhomogeneous eq.: $\dot{\xi} = \frac{eE_y}{m\omega}$

General solution of homogeneous eq.: $\dot{\xi} = a e^{-i\omega t}$ $(a \in \mathbb{C})$

(1) $\rightarrow \underline{\underline{\dot{\xi}(t) = a e^{-i\omega t} + eE_y/m\omega}}$ is the most general solution of $(*)$.

c) With $a = be^{i\alpha}$, $b, \alpha \in \mathbb{R}$

$\rightarrow \dot{\xi} = be^{-i(\omega-\alpha)t} + eE_y/(m\omega)$

$\rightarrow \alpha$ just shifts the zero of time $\rightarrow \underline{\underline{\alpha = 0}}$ w.l.o.g

$\rightsquigarrow \quad \dot x + i\dot y = b\cos\omega t - i b\sin\omega t + eE_y/m\omega$

$\rightsquigarrow \quad \boxed{\begin{array}{l} \dot x = b\cos\omega t + eE_y/m\omega \\ \dot y = -b\sin\omega t \end{array}} \quad (**)$

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$\rightsquigarrow \quad \langle \dot y\rangle = 0 \;, \quad \langle \dot x\rangle = eE_y/m\omega = cE_y/B$ time-averaged velocity

$= \frac{cE_y B}{B^2} = c(\vec E\times\vec B)_x/B^2$

in general:

0 $\quad \langle \vec v\rangle = \frac{c}{B^2}\,\vec E\times\vec B$ drift velocity

condition for $v \ll c$: $\quad E_y/B \ll 1$ necessary and sufficient condition for non-relativistic approximation

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d) Choose $x(t=0) = 0 = y(t=0)$ w.l.o.g.

$(**) \rightsquigarrow \quad \boxed{\begin{array}{l} x(t) = \frac{b}{\omega}\sin\omega t + \frac{cE_y}{B}t \\ y(t) = \frac{b}{\omega}(\cos\omega t - 1) \end{array}}$

0

$\rightsquigarrow$ The path is a trochoid.

To visualize it, put $\omega = 1$ and define $C = cE_y/B$.

$\rightsquigarrow \quad \boxed{\begin{array}{l} x(t) = b\sin t + Ct \\ y(t) = b(\cos t - 1) \end{array}}$ This is the projection of the path onto the x-y plane

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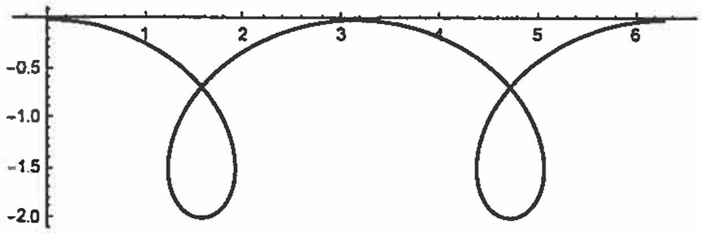
For C < b the trochoid has loops :

In[21]:=
```
b = 1;
c = 0.5;
x[t_] := b Sin[t] + c t
y[t_] := b (Cos[t] - 1)
ParametricPlot[{x[t], y[t]}, {t, 0, 4 Pi}]
```

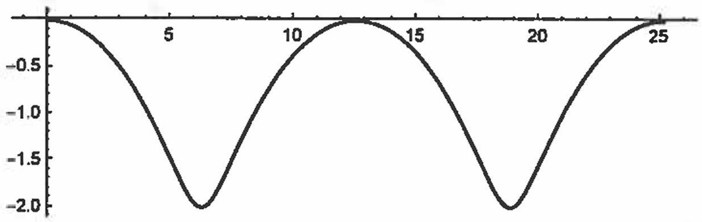
Out[25]=



For C > b it does not :

In[86]:=
```
b = 1;
c = 2;
x[t_] := b Sin[t] + c t
y[t_] := b (Cos[t] - 1)
ParametricPlot[{x[t], y[t]}, {t, 0, 4 Pi}, AspectRatio → 0.3]
```

Out[90]=



And for C = b it degenerates into a cycloid :

In[91]:=
```
b = 1;
c = 1;
x[t_] := b Sin[t] + c t
y[t_] := b (Cos[t] - 1)
ParametricPlot[{x[t], y[t]}, {t, 0, 4 Pi}, AspectRatio → 0.3]
```

Out[95]=