0.3.3. **Relativistic motion in parallel electric and magnetic fields**

Consider a relativistic charged particle (mass m, charge e) in parallel homogeneous electric and magnetic fields $\boldsymbol{E} = (0, 0, E)$, $\boldsymbol{B} = (0, 0, B)$.

a) Show that the equation of motion for the z-component of the momentum p_z decouples from p_x and p_y, and that the momentum perpendicular to the z-axis is a constant of motion: $p_x^2 + p_y^2 \equiv p_\perp^2 = \text{const.}$

b) Choose the zero of time such that $p_z(t = 0) = 0$, and show that with a suitable chosen origin the z-component of the particle's position can be written

$$z(t) = \frac{1}{eE} \sqrt{T_0^2 + c^2 e^2 E^2 t^2}$$

where T_0 is the kinetic energy (i.e., the energy of the particle without the potential energy due to the fields) at time $t = 0$.

hint: Recall Einstein's law of falling bodies, ch. 0 §3.3.

c) Introduce a parameter φ via $d\varphi/dt = ceB/T(t)$, with $T(t)$ the time-dependent kinetic energy. Show that the orbit of the particle can be represented in the parametric form

$$x = \frac{cp_\perp}{eB} \sin\varphi \quad , \quad y = \frac{cp_\perp}{eB} \cos\varphi \quad , \quad z = \frac{T_0}{eE} \cosh(E\varphi/B)$$

and explicitly find the relation between φ and t.

hint: Consider $\pi := p_x + ip_y$ and note that $|\pi| = p_\perp = \text{const.}$ by the result of part a).

d) Describe and visualize the orbit, and discuss the motion in the limits of large and small times.

(14 points)

0.3.3.) a) The lagrangian is

$$L = L_0 + eEz + \frac{e}{c}\vec{v}\cdot\vec{A}$$

with $L_0 = -mc^2\sqrt{1-v^2/c^2}$

and $\vec{A} = \frac{1}{2}\begin{pmatrix} -By \\ Bx \\ 0 \end{pmatrix} \rightarrow \vec{\nabla}\times\vec{A} = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}$ ✓

$\rightarrow \partial L/\partial z = eE$ independent of B

1 $\rightarrow \boxed{\dot{p}_z = eE}$ decoupled from p_x, p_y

$\rightarrow \underline{p_z(t) = eEt}$ (with $t=0$ chosen such that $p_z(t=0)=0$)

The forces in x and y-direction are

$$F_x = \frac{e}{c}(\vec{v}\times\vec{B})_x = \frac{eB}{c}v_y \;, \quad F_y = \frac{e}{c}(\vec{v}\times\vec{B})_y = -\frac{eB}{c}v_x$$

1 $\rightarrow \boxed{\dot{p}_x = \frac{eB}{c}v_y \;, \quad \dot{p}_y = -\frac{eB}{c}v_x}$ (*)

where $\vec{p} = (p_x, p_y, p_z) = \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}$ is the momentum

$$\rightarrow \underline{\underline{\frac{d}{dt}(p_x^2+p_y^2)}} = 2(p_x\dot{p}_x + p_y\dot{p}_y) = \frac{2meB}{c}\frac{1}{\sqrt{1-v^2/c^2}}(v_xv_y - v_yv_x) = \underline{\underline{0}}$$

1 $\rightarrow \underline{\underline{p_x^2+p_y^2 =: p_\perp^2 = \text{const}}}$

b) The kinetic energy is

$$T = \vec{v}\frac{\partial L_0}{\partial \vec{v}} - L_0 = \frac{mc^2}{\sqrt{1-v^2/c^2}} = \sqrt{m^2c^4+c^2p^2}$$

$$= \sqrt{m^2c^4+c^2p_\perp^2+c^2p_z^2} = \sqrt{T_0^2+c^2e^2E^2t^2}$$

1 where $T_0 = \sqrt{m^2c^4+c^2p_\perp^2} = T(t=0)$

But $\vec{v} = \frac{\vec{p}}{\sqrt{m^2+p^2/c^2}} = \frac{c\vec{p}}{\sqrt{m^2c^2+p^2}} = \frac{c^2\vec{p}}{T}$ (**)

(1) $\to \dot{z} = \frac{c^2 p_z}{T} = \frac{c^2 eEt}{\sqrt{T_0^2+c^2e^2E^2t^2}}$

$$\to \underline{\underline{z(t)}} = z_0 + c^2eE\int_0^t d\tau\, \frac{\tau}{\sqrt{T_0^2+c^2e^2E^2\tau^2}} = z_0 + \frac{c^2eE}{c^2e^2E^2}\,\frac{1}{2}\int_0^{ceEt} \frac{dx}{\sqrt{T_0^2+x}}$$

$$= z_0 + \frac{1}{eE}\,\frac{1}{2}\int_0^{ceEt/T_0} \frac{dx}{\sqrt{1+x}} = z_0 + \frac{T_0}{eE}\sqrt{1+c^2e^2E^2t^2/T_0^2} - \frac{T_0}{eE}$$

$$\underline{\underline{= \frac{1}{eE}\sqrt{T_0^2+c^2e^2E^2t^2}}} \quad \text{with} \quad \underline{z_0 = T_0/eE}$$

(1) cf PHYS 611 Problem 21 ✓

c) Define $\tilde{\pi} = p_x + i p_y$

$$(*) \to \dot{\tilde{\pi}} = \frac{eB}{c}(-i)(v_x+iv_y) \overset{(**)}{=} \frac{eB}{c}(-i)\frac{c^2}{T}\tilde{\pi} = -i\,\frac{eBc}{T}\,\tilde{\pi}$$

Define φ by $\frac{eBc}{T}dt = d\varphi$

(1) $\to \frac{d\tilde{\pi}}{d\varphi} = -i\tilde{\pi} \to \underline{\tilde{\pi} = p_\perp e^{-i\varphi}}$

$$\to \tilde{\pi} = \underline{p_\perp e^{-i\varphi}} \overset{(**)}{=} \frac{T}{c^2}(v_x+iv_y) = \frac{T}{c^2}\frac{d}{dt}(x+iy) = \frac{T}{c^2}\frac{eBc}{T}\frac{d}{d\varphi}(x+iy)$$

$$= \frac{eB}{c}\frac{d}{d\varphi}(x+iy)$$

$$\to \frac{dx}{d\varphi} = \frac{cp_\perp}{eB}\cos\varphi\,, \quad \frac{dy}{d\varphi} = -\frac{cp_\perp}{eB}\sin\varphi$$

(1) $\to \boxed{x = \frac{cp_\perp}{eB}\sin\varphi\,, \quad y = \frac{cp_\perp}{eB}\cos\varphi}$ with a suitable choice of origin

So that $\frac{d\varphi}{dt} = \frac{eBc}{T} = \frac{eBc}{\sqrt{T_0^2+c^2e^2E^2t^2}}$

$$\rightsquigarrow \quad \varphi = \frac{B}{E}\,\mathrm{arsinh}\left(\frac{ceEt}{T_0}\right)$$

check: $\frac{d\varphi}{dt} = \frac{B}{E}\,\frac{ceE}{T_0}\,\frac{1}{\sqrt{1+\left(\frac{ceEt}{T_0}\right)^2}} = \frac{eBc}{\sqrt{T_0^2+c^2e^2E^2t^2}}$ ✓

$$\rightsquigarrow \quad \cosh\frac{E\varphi}{B} = \cosh\,\mathrm{arsinh}\left(\frac{ceEt}{T_0}\right) = \sqrt{1+c^2e^2E^2t^2/T_0^2} = T/T_0$$

while b) $\rightsquigarrow$ $z = T/ceE$

$$\rightsquigarrow \quad \underline{\underline{z = \frac{T_0}{cE}\cosh\frac{E\varphi}{B}}}$$

1

Now we have the orbit in a parametric representation:

$$\boxed{x = \frac{cp_\perp}{eB}\sin\varphi\,, \quad y = \frac{cp_\perp}{eB}\cos\varphi\,, \quad z = \frac{T_0}{eE}\cosh\frac{E\varphi}{B}}$$

where φ is related to t via

$$\boxed{\varphi = \frac{B}{E}\,\mathrm{arsinh}\,\frac{ceEt}{T_0}} \quad \text{or} \quad \boxed{t = \frac{T_0}{ceE}\sinh\frac{E\varphi}{B}}$$

1

d). The orbit is a helix whose pitch increases exponentially with increasing angle. Choosing the unit of length such that $r_\perp := cp_\perp/eB = 1$ we have

$$\boxed{x = \sin\varphi\,,\ y = \cos\varphi\,,\ z = z_0\cosh(E\varphi/B)} \quad \text{with } z_0 = \frac{T_0 B}{cp_\perp E}$$

1

Here is an example for $z_0 = 1$, $E/B = 0.1$:

In[198]:=
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z0 = 1
EB = 0.1
x[phi_] := Sin[phi]
y[phi_] := Cos[phi]
z[phi_] := z0 Cosh[EB phi]
ParametricPlot3D[{x[phi], y[phi], z[phi]}, {phi, 0, 8 Pi}]
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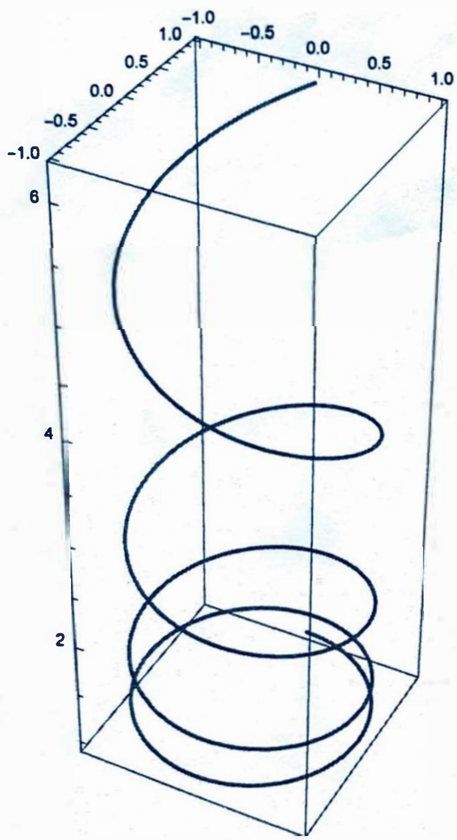
Out[198]= 1

Out[199]= 0.1

Out[203]=



1

For $t \gg \frac{T_0}{ceE}$ we have $T \to ceEt$

$\to \dot{\varphi} = \frac{ceB}{T} \to \frac{ceB}{ceE}\frac{1}{t} = \frac{B}{E}\frac{1}{t} \to 0$ the angular velocity goes to zero

1

$z(t) \to ct \to \dot{z} \to c$ the velocity in z-direction approaches c

For $t \ll \frac{T_0}{ceE}$ we have $\dot{\varphi} = eBc/T_0 + O(t^2)$ const. angular velocity

and $z(t) = z_0 + \frac{1}{2}\frac{c^2 e}{T_0}t^2 + O(t^4)$ Galilean result

1