1.1.2. **Ginzburg-Landau theory**

Ginzburg and Landau postulated that superconductivity can be described by an action (which is NOT Lorentz invariant)

$$S_{\text{GL}} = \int d\boldsymbol{x} \Big[r\,\big|\phi(\boldsymbol{x})\big|^2 + c\,\big|\big[\nabla - iq\boldsymbol{A}(\boldsymbol{x})\big]\phi(\boldsymbol{x})\big|^2 + u\,|\phi(\boldsymbol{x})|^4 + \frac{1}{16\pi\mu}\,F_{ij}(\boldsymbol{x})\,F^{ij}(\boldsymbol{x})\Big]$$

Here $\boldsymbol{x} \in \mathbb{R}^3$, and $\phi(\boldsymbol{x})$ is a complex-valued field that describes the superconducting matter, $\boldsymbol{A}$ is the Euclidian vector field that comprises the spatial components of the 4-vector $A^\mu = (A^0, \boldsymbol{A})$, and $F_{ij} = \partial_i A_j - \partial_j A_i$ $(i, j = 1, 2, 3)$. μ and q are coupling constants that characterize the vector potential and its coupling to the matter, and r, c and u are further parameters of the theory.

a) Find the coupled differential equations (known as Ginzburg-Landau equations) whose solutions extremize this action by considering the functional derivatives of S_{GL} with respect to all independent fields. (See Problem 0.2.4. You may want to double check against what you get from the Landau-Lifshitz method we used in class.)

b) Show that this theory is invariant under gauge transformations $\phi(x) \to \phi(\boldsymbol{x})\,e^{iq\lambda(\boldsymbol{x})}$, $\boldsymbol{A}(\boldsymbol{x}) \to \boldsymbol{A}(\boldsymbol{x}) + \boldsymbol{\nabla}\lambda(\boldsymbol{x})$.

c) Show that the Lorentz-invariant Lagrangian density for a massive scalar field, Problem 0.2.5, can be made gauge invariant by coupling $\phi(x)$ to the electromagnetic vector potential $A^\mu(x)$.

   hint: Replace the 4-gradient ∂_μ by $D_\mu = \partial_\mu - iqA_\mu$ and add the Maxwell Lagrangian.

   note: If we had never heard of the electromagnetic potential, insisting on gauge invariance would force us to invent it!

(7 points)

1.1.2.)

$$\boxed{S_{GL} = \int d\vec{x}\left[r|\phi(\vec{x})|^2 + c\left|(\vec{\nabla}-ig\vec{A}(\vec{x}))\phi(\vec{x})\right|^2 + u|\phi(\vec{x})|^4 + \frac{1}{16\pi\mu}F_{ij}(\vec{x})F^{ij}(\vec{x})\right]}$$

c) $0 \overset{!}{=} \frac{\delta S_{GL}}{\delta\phi^*(\vec{x})} = r\phi(\vec{x}) + 2u\phi(\vec{x})|\phi(\vec{x})|^2 + c\frac{\delta}{\delta\phi^*(\vec{x})}\int d\vec{x}\,(\vec{\nabla}-ig\vec{A})\phi\,(\vec{\nabla}+ig\vec{A})\phi^*$

$= r\phi + 2u\phi|\phi|^2 - c\vec{\nabla}(\vec{\nabla}-ig\vec{A})\phi + cig\vec{A}(\vec{\nabla}-ig\vec{A})\phi$

1 $= r\phi + 2u\phi|\phi|^2 - c(\vec{\nabla}-ig\vec{A})^2\phi$

$0 \overset{!}{=} \frac{\delta S_{GL}}{\delta\vec{A}(\vec{x})} = c(-ig)\phi(\vec{\nabla}+ig\vec{A})\phi^* + cig\left((\vec{\nabla}-ig\vec{A})\phi\right)\phi^*$

1 $+ \frac{1}{16\pi\mu}\frac{\delta}{\delta\vec{A}(\vec{x})}\int d\vec{y}\,F_{ij}(\vec{y})F^{ij}(\vec{y})$

$F_{ij}F^{ij} = (\partial_i A_j - \partial_j A_i)(\partial^i A^j - \partial^j A^i) = 2\varepsilon^{kij}\varepsilon_{klm}\partial_i A_j\partial^l A^m$

$= 2(\vec{\nabla}\times\vec{A})^2$

$\rightarrow \frac{\delta}{\delta\vec{A}(\vec{x})}\int d\vec{y}\,F_{ij}F^{ij} = -\frac{\delta}{\delta\vec{A}(\vec{x})}\int d\vec{y}\,\vec{A}(\vec{y})\cdot\left(\vec{\nabla}\times(\vec{\nabla}\times\vec{A}(\vec{y}))\right)$

1 $= -2\,\vec{\nabla}\times(\vec{\nabla}\times\vec{A}(\vec{x}))$

$= cig\,\phi^*(\vec{\nabla}-ig\vec{A})\phi + \text{c.c.} - \frac{1}{4\pi\mu}\vec{\nabla}\times(\vec{\nabla}\times\vec{A}(\vec{x}))$

$$\rightarrow \boxed{-c\left[\vec{\nabla}-ig\vec{A}(\vec{x})\right]^2\phi(\vec{x}) + \left[r + 2u|\phi(\vec{x})|^2\right]\phi(\vec{x}) = 0}$$

1 $$\boxed{\vec{\nabla}\times(\vec{\nabla}\times\vec{A}(\vec{x})) = 4\pi\mu c i g\,\phi^*(\vec{x})\left[\vec{\nabla}-ig\vec{A}(\vec{x})\right]\phi(\vec{x}) + \text{c.c.}}$$

GL eqs.

b) let $\phi(\bar{x}) \to \phi(\bar{x}) e^{ig\lambda(\bar{x})}$, $\vec{A}(\bar{x}) \to \vec{A}(\bar{x}) + \vec{\nabla}\lambda(\bar{x})$

$$\to \underline{|\phi(\bar{x})|^2 \to |\phi(\bar{x})|^2}$$

and $\underline{F_{ij}(\bar{x})} = \partial_i A_j - \partial_j A_i \to \partial_i (A_j + \cancel{\partial_j \lambda}) - \partial_j (A_i + \cancel{\partial_i \lambda}) = \underline{F_{ij}(\bar{x})}$

1 Finally,

$$(\vec{\nabla} - ig\vec{A})\phi \to (\vec{\nabla} - ig\vec{A} - ig\vec{\nabla}\lambda)\phi e^{ig\lambda} =$$

$$= (\vec{\nabla}\phi)e^{ig\lambda} + \cancel{ig(\vec{\nabla}\lambda)\phi e^{ig\lambda}} - ig\vec{A}\phi e^{ig\lambda} - \cancel{ig(\vec{\nabla}\lambda)\phi e^{ig\lambda}}$$

$$= e^{ig\lambda}(\vec{\nabla} - ig\vec{A})\phi$$

$$\to \underline{|(\vec{\nabla} - ig\vec{A})\phi|^2 \to |(\vec{\nabla} - ig\vec{A})\phi|^2}$$

1 $\to$ <u>S_{GL} is gauge invariant</u>

c) Modify $\mathcal{L}$ from Problem 0.2.5 b) to read

$$\boxed{\mathcal{L} = (D_\mu \phi(x))(D^\mu \phi(x))^* - m^2 |\phi(x)|^2 - \frac{1}{16\pi} F_{\mu\nu}(x) F^{\mu\nu}(x)} \quad (*)$$

where $D_\mu = \partial_\mu - igA_\mu$

let $\phi(x) \to \phi(x) e^{ig\lambda(x)}$, $A^\mu(x) \to A^\mu(x) + \partial^\mu \lambda(x)$

$\to |\phi|^2 \to |\phi|^2$ and $F_{\mu\nu}F^{\mu\nu} \to F_{\mu\nu}F^{\mu\nu}$

$$D_\mu \phi \to (\partial_\mu - igA_\mu - ig\partial_\mu\lambda)\phi e^{ig\lambda}$$

$$= e^{ig\lambda}(\partial_\mu + \cancel{ig\partial_\mu\lambda} - igA_\mu - \cancel{ig\partial_\mu\lambda})\phi = e^{ig\lambda} D_\mu \phi$$

$$\to \underline{(D_\mu \phi)(D^\mu \phi)^* \to (D_\mu \phi)(D^\mu \phi^*)}$$

1 $\to$ <u>$(*)$ is gauge invariant</u>