2.2.2. **Spherically symmetric charge distributions**

Consider a spherically symmetric static charge distribution (in spherical coordinates): $\rho(\boldsymbol{x}) = \rho(r)$.

a) Express the electric field in terms of a one-dimensional integral over $\rho(r)$, and the electrostatic potential by a one-dimensional integral over the field.

   hint: Make an *ansatz* for a purely radial field, $\boldsymbol{E}(\boldsymbol{x}) = E(r)\,\hat{e}_r$, and integrate Gauss's law over a spherical volume.

Explicitly calculate and plot the field $\boldsymbol{E}(\boldsymbol{x})$ and the potential $\varphi(\boldsymbol{x})$ for

b) a homogeneously charged sphere

$$\rho(\boldsymbol{x}) = \begin{cases} \rho_0 & \text{if } \; r \leq r_0 \\ 0 & \text{if } \; r > r_0 \; . \end{cases}$$

c) a homogeneously charged spherical shell

$$\rho(\boldsymbol{x}) = \sigma_0 \, \delta(r - r_0) \; .$$

(8 points)

2.2.2-1 a) Consider Gauss's law $\vec\nabla\cdot\vec E = 4\pi\rho$

and integrate over a sphere with volume V:

$$\int_V d\vec x\, \vec\nabla\cdot\vec E = \int_{(V)} d\vec S\cdot\vec E = 4\pi\int_V d\vec x\,\rho$$

let $\rho(\vec x)$ be spherically symmetric, $\rho(\vec x)=\rho(r)$, and make an

ansatz: $\boxed{\vec E(\vec x) = E(r)\,\hat e_r}$ $\quad \hat e_r = \vec x/|\vec x|$

$$4\pi r^2 E(r) = 4\pi\cdot 4\pi\int_0^r dr'\, r'^2\rho(r')$$

$$\boxed{E(r) = \frac{4\pi}{r^2}\int_0^r dr'\, r'^2\rho(r')}$$

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For the potential, write $\vec E(\vec x) = -\vec\nabla\varphi(\vec x)$

spherical symmetry $\rightsquigarrow$ $\vec\nabla\varphi = \partial_r\varphi\,\hat r$

$\rightsquigarrow$ $E(r) = -\partial_r\varphi(r)$

$\rightsquigarrow$ $\varphi(r) = -\int_\infty^r dr'\, E(r')$ $\quad$ if we choose $\varphi(r=\infty)=0$

$\rightsquigarrow$ $\varphi(r) = \int_r^\infty dr'\, E(r')$

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$$E(r) = \frac{4\pi}{r^2}\int_0^r dr'\, r'^2\rho_0\,\Theta(r'<r_0)$$

1st case: $r<r_0$ $\quad \underline{\underline{E(r)}} = \frac{4\pi\rho_0}{r^2}\int_0^r dx\, x^2 = \frac{4\pi\rho_0}{r^2}\frac{1}{3}r^3 = \frac{4\pi}{3}\rho_0 r$

$$= \frac{4\pi}{3}r_0^3\rho_0\frac{r}{r_0^3} = \frac{Qr}{r_0^3}$$

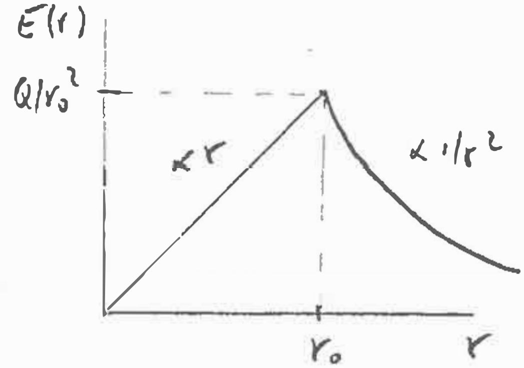
$Q = \frac{4\pi}{3}r_0^3\rho_0$

= total charge

2nd case: $r > r_0$ $\quad E(r) = \frac{4\pi}{r^2}\int_0^{r_0} dr'\, r'^2 \rho_0 = \frac{4\pi}{r^2}\rho_0 \frac{1}{3} r_0^3 = Q/r^2$

$$\rightarrow \quad \boxed{E(r) = \begin{cases} Qr/r_0^3 & \text{for } r \le r_0 \\ Q/r^2 & \text{for } r > r_0 \end{cases}}$$

$$\boxed{\vec{E}(\vec{x}) = E(r)\,\hat{e}_r}$$



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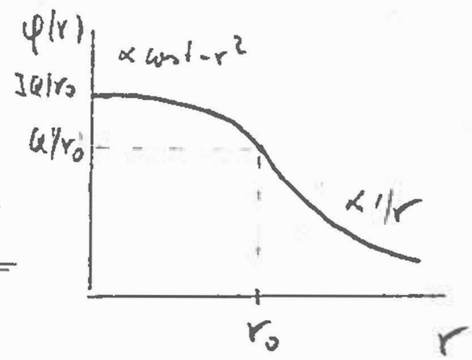
Now the potential:

1st case: $r < r_0$: $\quad \varphi(r) = \int_r^{r_0} dr' \frac{Qr'}{r_0^3} + \int_{r_0}^{\infty} dr' \frac{Q}{r'^2} = \frac{Q}{r_0^3}\frac{1}{2}(r_0^2 - r^2) + \frac{Q}{r_0}$

$$= \frac{Q}{2r_0^3}(3r_0^2 - r^2)$$



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2nd case: $r > r_0$ $\quad \varphi(r) = \int_r^{\infty} dr' \frac{Q}{r'^2} = \frac{Q}{r}$

$$\boxed{\varphi(r) = \begin{cases} \frac{Q}{2r_0^3}(3r_0^2 - r^2) & \text{for } r < r_0 \\ Q/r & \text{for } r \ge r_0 \end{cases}}$$

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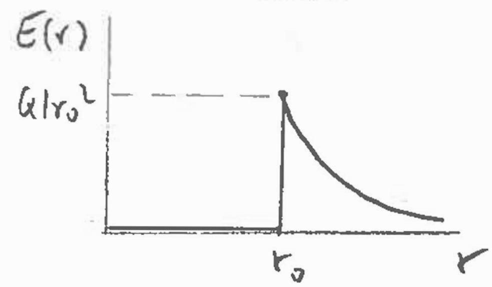
c) electric field: $r < r_0$ $\quad E(r) = 0$

$r > r_0$ $\quad E(r) = \frac{4\pi}{r^2}\sigma_0 r_0^2 = \frac{Q}{r^2}$

$$\boxed{E(r) = \begin{cases} 0 & \text{for } r < r_0 \\ Q/r^2 & \text{for } r > r_0 \end{cases}}$$



with $Q = 4\pi r_0^2 \sigma_0$ = total charge

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For $r > r_0$, $E(r)$ is the same as for the homogeneous sphere!

potential: $r < r_0$ $\quad \varphi(r) = \int_{r_0}^{\infty} dr' \frac{Q}{r'^2} = Q/r_0$

$r > r_0$ $\quad \varphi(r) = \int_{r}^{\infty} dr' \frac{Q}{r'^2} = Q/r$

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$$\boxed{\varphi(r) = \begin{cases} Q/r_0 & \text{for } r < r_0 \\ Q/r & \text{for } r > r_0 \end{cases}}$$