

2.2.3

Electrostatics in d dimensions (12 pts)

Consider the third Maxwell equation in d dimensions:

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = S_d \rho(\mathbf{x})$$

with the electric field \mathbf{E} a d -vector, and S_d the area of the $(d-1)$ -sphere: $S_{2n} = 2\pi^n/(n-1)!$ and $S_{2n+1} = 2^{2n+1}n!\pi^n/(2n)!$ for even and odd dimensions, respectively. Define a scalar potential $\varphi(\mathbf{x})$ in analogy to the $3-d$ case, such that

$$\mathbf{E}(\mathbf{x}) = -\nabla\varphi(\mathbf{x})$$

and consider Poisson's equation

$$\nabla^2\varphi(\mathbf{x}) = -S_d \rho(\mathbf{x})$$

note: Here we consider a generalization of electrostatics to d -dimensional space, NOT a d -dimensional charge distribution embedded in 3-dimensional space.

a) Show that the Green function $G_d(\mathbf{x})$ function for Poisson's equation, i.e., the solution of

$$\nabla^2 G_d(\mathbf{x}) = -S_d \delta(\mathbf{x})$$

is given by

$$G_d(\mathbf{x}) = \frac{1}{d-2} \frac{1}{|\mathbf{x}|^{d-2}}$$

for all $d \neq 2$, and by

$$G_2(\mathbf{x}) = \ln(1/|\mathbf{x}|)$$

for $d = 2$.

hint: For $d = 1$, differentiate directly, using $d \operatorname{sgn} x/dx = 2\delta(x)$. For $d \geq 2$, show that $G_d(\mathbf{x})$ is a harmonic function for all $\mathbf{x} \neq 0$, then integrate $\nabla^2 G_d$ over a hypersphere around the origin and use Gauss's law.

b) Calculate and plot the potential φ and the field \mathbf{E} for $d = 2$ for the case of a homogeneously charged disk, $\rho(\mathbf{x}) = \rho_0 \Theta(r_0 - |\mathbf{x}|)$.

hint: It is easiest to proceed as in the $3-d$ case, see Problem 2.2.2.

note: This problem plays an important role in the theory of the Kosterlitz-Thouless transition, for which part of the 2016 Nobel prize in Physics was awarded.

c) The same for $d = 1$ for the case of a uniformly charged rod, $\rho(x) = \rho_0 \Theta(x_0^2/4 - x^2)$.

hint: Integrate Poisson's formula directly.

2.2.3) a)

wird $\nabla^2 G_d(\vec{x}) = -\int_{S_d} \delta(\vec{x})$

mit S_d die surface einer $(d-1)$ -Sphäre.

Proposition: $G_d(\vec{x}) = \frac{1}{d-2} \frac{1}{|\vec{x}|^{d-2}}$ für $d \neq 2$

$G_d(\vec{x}) = \ln(1/|\vec{x}|)$ für $d=2$

Proof: $d=1$ By direct differentiation: 610 Proben 16b)

$$\frac{d^2}{dx^2} (-1/|x|) = -\frac{d}{dx} \frac{1}{|x|} = -2\delta(x)$$

$$\rightarrow \frac{d^2}{dx^2} G_{d=1}(x) = \frac{d^2}{dx^2} (-1/|x|) = -2\delta(x) =$$

$$= -\int_{S_{d=1}} \delta(x) \quad \checkmark$$

$d=2$ $r=|\vec{x}|$

$$\partial_i \partial_i \ln|\vec{x}| = \partial_i \frac{x_i}{r^2} = \frac{r^2 \delta_{ij} - x_i x_j}{r^4} \frac{2x_i}{r} = \frac{r^2 \delta_{ij} - 2x_i x_j}{r^4}$$

$$\rightarrow \nabla^2 \ln|\vec{x}| = \partial_i \partial_i \ln|\vec{x}| = (2-2) \frac{1}{r^2} = 0 \quad \forall r \neq 0$$

$\rightarrow \ln|\vec{x}|$ is a harmonic fct. $\forall \vec{x} \neq 0$

Now integrate over a circle C_0 radius r_0 :

$$\int_{C_0} d^2x \nabla^2 \ln|\vec{x}| = \int_{C_0} d^2x \nabla \cdot (\nabla \ln|\vec{x}|) \stackrel{\text{Gauss}}{=} \int_{(C_0)} d\vec{\sigma} \cdot \nabla \ln|\vec{x}|$$

$$= \int_0^{2\pi} d\varphi r_0 \frac{\vec{x} \cdot \vec{x}}{r^2} \Big|_{r_0} = 2\pi$$

$$\rightarrow \nabla^2 \ln|\vec{x}| = 2\pi \delta(\vec{x}) = \int_{S_{d=2}} \delta(\vec{x})$$

$$\rightarrow \underline{\underline{G_{d=2}(\vec{x}) = -\ln|\vec{x}| = \ln(1/|\vec{x}|)}}$$

$$d > 2: \quad \partial_i \partial_j \frac{1}{|\bar{x}|^{d-2}} = -\frac{d-2}{r} \partial_i \frac{\lambda x_j}{|\bar{x}|^d} = -(d-2) \left(\frac{\delta_{ij}}{|\bar{x}|^d} - \frac{d}{r} \frac{\lambda x_i x_j}{|\bar{x}|^{d+2}} \right)$$

$$= -(d-2) \frac{r^2 \delta_{ij} - d x_i x_j}{r^{d+2}}$$

$$\Rightarrow \nabla^2 \frac{1}{|\bar{x}|^{d-2}} = -(d-2)(d-d) \frac{1}{r^d} = 0 \quad \forall r \neq 0$$

Integrate over a hypersphere S_0^d with radius r_0 :

$$\int_{S_0^d} d^d x \nabla^2 \frac{1}{|\bar{x}|^{d-2}} = \int_{(S_0^d)} d\vec{\sigma} \cdot \nabla \frac{1}{|\bar{x}|^{d-2}} = \int_{S_0^d} r_0^{d-1} (-) (d-2) \frac{\bar{x} \cdot \bar{x}}{r_0^d}$$

$$= - (d-2) \int_{S_0^d} d\vec{\sigma}$$

$$\nabla^2 \frac{1}{|\bar{x}|^{d-2}} = - (d-2) \int_{S_0^d} \delta(\bar{x})$$

$$\underline{G_{d>2}(\bar{x})} = \frac{1}{d-2} \frac{1}{|\bar{x}|^{d-2}}$$

(1)

b) It is easiest to start with the field. Gauss's law in 2-d reads

$$\vec{\nabla} \cdot \vec{E}(\vec{x}) = \lambda_0 g(\vec{x})$$

and proceeding as in Problem 2.22 we have

$$\lambda_0 r E(r) = \lambda_0 \cdot \lambda_0 \int_0^r dr' r' g(r')$$

for a charge distribution $g(\vec{x}) = g(r)$ and $\vec{E}(\vec{x}) = E(r) \hat{e}_r$.

$$E(r) = \frac{\lambda_0}{r} \int_0^r dr' r' g(r')$$

(1)

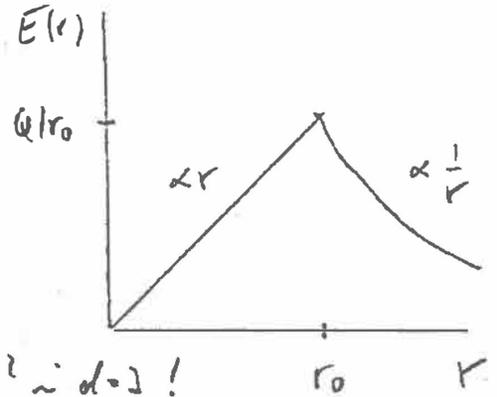
Homogeneous charged disk: $g(r) = g_0 \Theta(r_0 - r)$

1st con: $r < r_0$ $E(r) = \frac{\sigma}{r} \int_0^r dr' r' \rho_0 = \frac{\sigma \rho_0}{r} \frac{1}{2} r^2 = \sigma \rho_0 r$
 $= \frac{Q}{r_0^2} r$ with $Q = \sigma r_0^2 \rho_0$ total charge

2nd con: $r > r_0$ $E(r) = \frac{\sigma}{r} \rho_0 \frac{1}{2} r_0^2 = \frac{Q}{r}$

$\vec{E}(\vec{x}) = E(r) \hat{e}_r$

$$E(r) = \begin{cases} Q r / r_0^2 & \text{für } r < r_0 \\ Q / r & \text{für } r > r_0 \end{cases}$$



Field falls off as $1/r$, as opposed to $1/r^2$ in $d=3$!

Now the potential: $\vec{E}(\vec{x}) = -\vec{\nabla} \phi(\vec{x}) = -\partial_r \phi(r) \hat{e}_r$

$\rightarrow E(r) = -\partial_r \phi(r)$

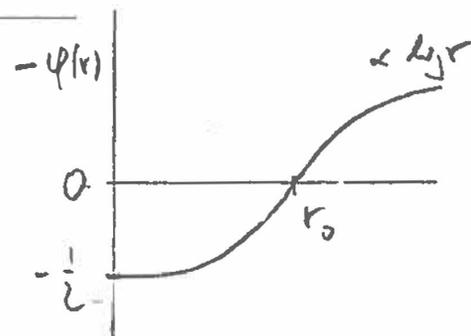
$\rightarrow \phi(r) = - \int_{\infty}^r dr' E(r')$

$\rightarrow \phi(r) = - \int_{r_0}^r dr' E(r')$ with the choice $\phi(r=r_0) = 0$

1st con: $r < r_0$ $-\phi(r) = - \int_{r_0}^r dr' \frac{Q r'}{r_0^2} = + \frac{Q}{2 r_0^2} (r_0^2 - r^2) = \frac{+Q}{2} \left(\frac{r^2}{r_0^2} - 1 \right)$

2nd con: $r > r_0$ $-\phi(r) = \int_{r_0}^r dr' \frac{Q}{r'} = Q \ln(r/r_0)$

$$-\phi(r) = Q \cdot \begin{cases} \frac{1}{2} \left(\frac{r^2}{r_0^2} - 1 \right) & \text{für } r < r_0 \\ \ln(r/r_0) & \text{für } r > r_0 \end{cases}$$



mit: This is minus ϕ !

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her sketches

c) In 1-d it is easiest to integrate Poisson's formula directly:

$$\varphi(x) = \int_{-x_0/2}^{x_0/2} G_{d=1}(x-y) \rho(y) dy = -\int_{-x_0/2}^{x_0/2} |x-y| \rho_0 \Theta(x_0^2/4 - y^2) dy$$

$$= -\rho_0 \int_{-x_0/2}^{x_0/2} |x-y| dy = \varphi(-x)$$

Let $x \geq 0$.

1st case: $x < x_0/2$ $\varphi(x) = -\rho_0 \int_{-x_0/2}^x (x-y) dy + \rho_0 \int_x^{x_0/2} (x-y) dy$

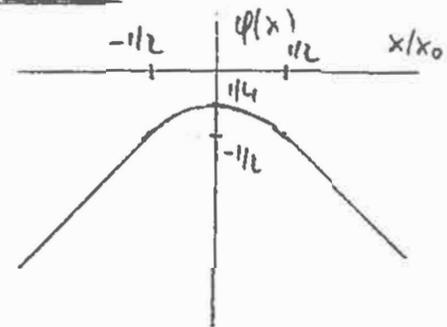
$$= -\rho_0 \left[x \left(x + \frac{x_0}{2} \right) - \frac{1}{2} \left(x^2 - \frac{1}{4} x_0^2 \right) \right] + \rho_0 \left[x \left(\frac{x_0}{2} - x \right) - \frac{1}{2} \left(\frac{x_0^2}{4} - x^2 \right) \right]$$

$$= -\rho_0 \left(x^2 + \frac{1}{4} x_0^2 \right) = -$$

2nd case: $x > x_0/2$ $\varphi(x) = -\rho_0 \int_{-x_0/2}^{x_0/2} (x-y) dy = -\rho_0 x x_0 = -\int_0^{x_0} x_0 x$

$= -Qx$ with $Q = \rho_0 x_0$ total charge

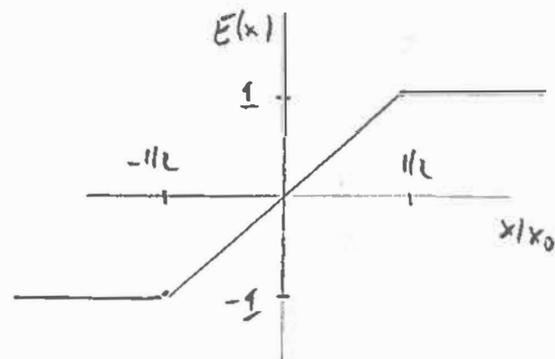
$$\varphi(x) = -Qx_0 \times \begin{cases} \frac{x^2}{x_0^2} + \frac{1}{4} & \text{for } |x| < x_0/2 \\ |x|/x_0 & \text{for } |x| > x_0/2 \end{cases}$$



Now the field:

$$E(x) = -\partial_x \varphi(x) =$$

$$E(x) = Q \times \begin{cases} 2x/x_0 & \text{for } |x| < x_0/2 \\ \text{sgn } x & \text{for } |x| > x_0/2 \end{cases}$$



Field does not fall off for $|x| \rightarrow \infty$!