2.3.5.) **Field due to distant charges** (6 pts)

Consider the electric field generated by a charge density $\rho(\boldsymbol{y})$ that vanishes inside a sphere with radius r_0: $\rho(\boldsymbol{y}) = 0$ for $|\boldsymbol{y}| \leq r_0$. Show that

a) If ρ is invariant under parity operations, $\rho(-\boldsymbol{y}) = \rho(\boldsymbol{y})$, then the electric field at the origin vanishes.

b) If $\rho(\boldsymbol{y})$ is invariant under rotations about the z-axis through multiples of an angle α with $|\alpha| < \pi$, then the field-gradient tensor at the origin has the form $\varphi_{ij}(\boldsymbol{x} = 0) = \begin{pmatrix} \varphi & 0 & 0 \\ 0 & \varphi & 0 \\ 0 & 0 & -2\varphi \end{pmatrix}$

c) If $\rho(\boldsymbol{y})$ has cubic symmetry, i.e., if $\rho(\boldsymbol{y})$ is invariant under rotations through $\pi/2$ about any of the three axes x, y, and z, then the field-gradient tensor at the origin vanishes.

2.3.5.) $\varphi(\vec{x}) = \int d\vec{\xi}\, \frac{\rho(\vec{\xi})}{|\vec{x}-\vec{\xi}|} = \varphi(\vec{x}=0) + \vec{x}\cdot\vec{\nabla}\varphi\Big|_{\vec{x}=0} + \frac{1}{2} x_i x_j \frac{\partial^2\varphi}{\partial x_i \partial x_j}\Big|_{\vec{x}=0} + \dots$

$$=: \varphi_0 - \vec{x}\cdot\vec{E} + \frac{1}{2} x_i x_j \varphi_{ij} + \dots$$

$$\equiv \varphi_0 + \varphi_1(\vec{x}) + \varphi_2(\vec{x}) + \dots$$

a) $\rho(\vec{\xi}) = \rho(-\vec{\xi}) \rightsquigarrow \varphi(-\vec{x}) = \int d\vec{\xi}\, \frac{\rho(\vec{\xi})}{|\vec{x}+\vec{\xi}|} = \int d\vec{\xi}\, \frac{\rho(-\vec{\xi})}{|\vec{x}-\vec{\xi}|} = \varphi(\vec{x})$

(1) $\rightsquigarrow$ All terms odd in $\vec{x}$ vanish, in particular $\underline{\vec{E} = 0}$

b) φ_{ij} is real symmetric $\rightsquigarrow$ $\exists$ coordinate system such that φ_{ij} is diagonal

(1)

$\varphi(\vec{x})$ obeys Laplace's eq. $\forall\ |\vec{x}| < r_0$

$\rightsquigarrow \sum_i \varphi_{ii} = 0$

$\rightsquigarrow \varphi_{ij}$ has the form $\varphi_{ij} = \begin{pmatrix} \varphi_+ + \varphi_- & 0 & 0 \\ 0 & \varphi_+ - \varphi_- & 0 \\ 0 & 0 & -2\varphi_+ \end{pmatrix}$

(1) where $\varphi_- = \frac{1}{2}(\varphi_{11} - \varphi_{22})$

$\rightsquigarrow \underline{\varphi_2(\vec{x})} = \frac{1}{2} r^2 \sin^2\vartheta \cos^2\varphi\, (\varphi_+ + \varphi_-)$
$+ \frac{1}{2} r^2 \sin^2\vartheta \sin^2\varphi\, (\varphi_+ - \varphi_-)$
$+ \frac{1}{2} r^2 \cos^2\vartheta\, (-2\varphi_+)$

(1) $= \underline{\frac{1}{2} r^2 \left[(1 - 3\cos^2\vartheta)\varphi_+ + \sin^2\vartheta \cos 2\varphi\, \varphi_- \right]}$

Rotational invariance of $\rho(\vec{\xi})$ implies rotational invariance of $\varphi(\vec{x})$, and in particular of $\varphi_2(\vec{x})$

$$\rightsquigarrow \underline{\varphi_2(r,\vartheta,\varphi+\alpha)} = \frac{1}{2}r^2\left[(1-3\cos^2\vartheta)\,\varphi_+ + \sin^2\vartheta\,\cos 2(\varphi+\alpha)\,\varphi_-\right]$$

$$\stackrel{!}{=} \underline{\varphi_2(r,\vartheta,\varphi)}$$

1 $$\rightsquigarrow \varphi_-\cos(2\varphi+2\alpha) = \varphi_-\cos 2\varphi \quad \rightsquigarrow \quad \underline{\underline{\varphi_- = 0}}$$

c) cubic symmetry $\rightsquigarrow$ $\rho(\vec{\xi})$ invariant under rotation through angle $\frac{\pi}{2}$ about any of the three axes x, y, z.

b) $\rightsquigarrow$ $\varphi_- = 0$ due to invariance under rotation about z-axis

Rotation about x or y $\rightsquigarrow$ $\vartheta \to \vartheta + \pi/2$

But invariance of $\rho(\vec{\xi})$ implies invariance of $\varphi(\vec{x})$

$$\rightsquigarrow \underline{\varphi_2(r,\vartheta+\tfrac{\pi}{2},\varphi)} = \frac{1}{2}r^2\varphi_+\left[1-3\cos^2(\vartheta+\pi/2)\right]$$

$$\stackrel{!}{=} \underline{\frac{1}{2}r^2\varphi_+\left[1-3\cos^2\vartheta\right]}$$

1 $$\rightsquigarrow \varphi_+\cos^2(\vartheta+\pi/2) = \varphi_+\cos^2\vartheta \quad \rightsquigarrow \quad \underline{\varphi_+ = 0} \quad \rightsquigarrow \quad \underline{\underline{\varphi_{ij} = 0}}$$