2.3.6. Electrostatic interaction I: Dipole in an external field

Three equal charges q are sitting on the corners of an isosceles triangle in the x-y plane, each at a distance a from the origin. At the center of the triangle sits an electric dipole $\boldsymbol{d}$. The positions of the charges and the dipole are fixed, but the dipole is free to rotate in the x-y plane. Calculate the electrostatic interaction energy of this system to dipolar order, and determine the equilibrium orientation of the dipole (i.e., the angle ϕ that minimizes the energy) as a function of the angle α. What happens in the special case of an equilateral triangle?
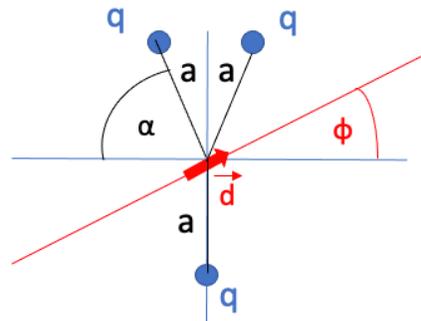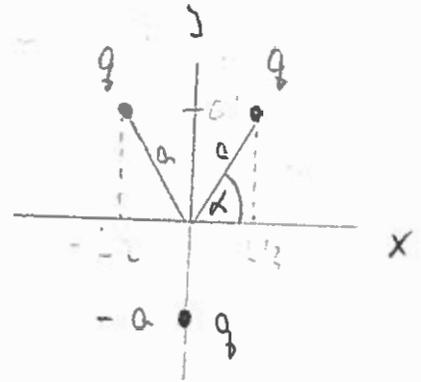

(5 points)

2.3.6.) Charges q located at

$$\vec{\zeta}^{(3)} = (0,-a) = a(0,-1)$$

$$\vec{\zeta}^{(1,2)} = a(\pm\cos\alpha, \sin\alpha)$$

Electrostatic potential:

$$\varphi(\vec{x}) = q \sum_{\lambda=1}^{3} \frac{1}{|\vec{x} - \vec{\zeta}^{(\lambda)}|}$$

$\rightarrow \varphi_0 = \varphi(\vec{x}=0) = q \sum_{\lambda=1}^{3} \frac{1}{|\vec{\zeta}^{(\lambda)}|} = \frac{3q}{a}$ potential at origin
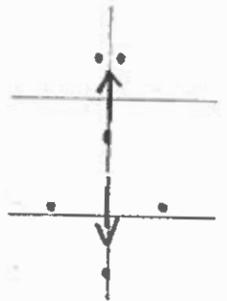
$\vec{E} = -\vec{\nabla}\big|_{\vec{x}=0} \varphi(\vec{x}) = q \sum_{\lambda=1}^{3} \frac{-\vec{\zeta}^{(\lambda)}}{|\vec{\zeta}^{(\lambda)}|^3} = -\frac{q}{a^2}\begin{pmatrix} 0 \\ 2\sin\alpha - 1 \end{pmatrix}$ field at origin

dipole: $\quad \vec{d} = d(\cos\varphi, \sin\varphi)$

electrostatic interaction energy:

$U = -\vec{E}\cdot\vec{d} = \frac{qd}{a^2}(2\sin\alpha - 1)\sin\varphi$

1st case: $\underline{qd > 0}$ $\varphi = \begin{cases} \pi/2 & \text{for } \alpha > \pi/6 \\ -3\pi/2 & \text{for } \alpha < \pi/6 \end{cases}$

2nd case: $\underline{qd < 0}$ $\varphi = \begin{cases} 3\pi/2 & \text{for } \alpha > \pi/6 \\ \pi/2 & \text{for } \alpha < \pi/6 \end{cases}$

special case: $\alpha = \frac{\pi}{6}$ (equilateral triangle) $U = 0 \; \forall \varphi$
   no preferred orientation