5 Elementary oscillations

5.1 Introduction

One of the oldest and best-known oscillating systems is the so-called **simple pendulum.** This consists of a light rigid rod of length l with a point mass at one end, the other end being pivoted at a fixed point O, so that the pendulum can swing freely in one particular vertical plane (Fig. 5.1).

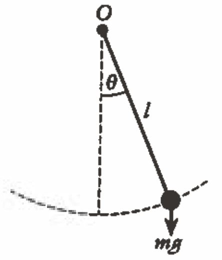

Fig. 5.1 The simple pendulum.

Galileo made careful observations of pendulums in about 1602, and found the period of oscillation T to be proportional to the square root of the length:

$$T \propto \sqrt{l}. \tag{5.1}$$

What really impressed him, however, was the way in which the oscillation period seemed to be independent of the amplitude, although it emerged subsequently that this is true only for *small* oscillations of the pendulum.

In order to set up the appropriate differential equation of motion, let θ (in radians) denote the angle between the pendulum and the downward vertical (Fig. 5.1). The bob has at any instant a velocity $l\,\mathrm{d}\theta/\mathrm{d}t$ in the direction of increasing θ, and its acceleration component in that direction is $l\,\mathrm{d}^2\theta/\mathrm{d}t^2$. The only force on the bob in that direction is a component $mg\sin\theta$ due to gravity, and this acts in the opposite sense, i.e. the direction of decreasing θ. So $ml\,\mathrm{d}^2\theta/\mathrm{d}t^2$ must be equal to $-mg\sin\theta$, i.e.

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}+\frac{g}{l}\sin\theta=0. \tag{5.2}$$

This is not easy to solve analytically, but if we confine attention to *small* swings of the pendulum we may use the approximation

$$\sin\theta \doteqdot \theta \qquad \text{for} \quad |\theta| \ll 1, \tag{5.3}$$

and we then have

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}+\frac{g}{l}\theta=0 \tag{5.4}$$

as the approximate equation of motion governing small-amplitude oscillations of a simple pendulum.

Suppose, then, that we draw the pendulum aside a small amount, to $\theta=\theta_0$, say, at $t=0$, and release it from rest. The solution of (5.4) which satisfies these initial conditions is

$$\theta=\theta_0\cos\left(\frac{g}{l}\right)^{1/2}t \tag{5.5}$$

(see (3.28)), so the pendulum swings to and fro with a period

$$T=2\pi\sqrt{\frac{l}{g}}. \tag{5.6}$$

This is proportional to $\sqrt{l}$ and independent of the amplitude θ_0, as Galileo observed in his experiments.

5.2 The linear oscillator

The kind of simplification we saw in moving from (5.2) to (5.4)—called **linearization** of the original equation of motion—is typical of virtually any dynamical system when we restrict attention to *small displacements about a point of (stable) equilibrium.*

To see why this should be so, consider the point mass m in Fig. 5.2, which moves to and fro under the action of a spring. Let $x=0$ denote the equilibrium point, at which the spring is neither extended nor compressed, and let the force exerted by the spring be $F(x)$ in the negative x-direction. In

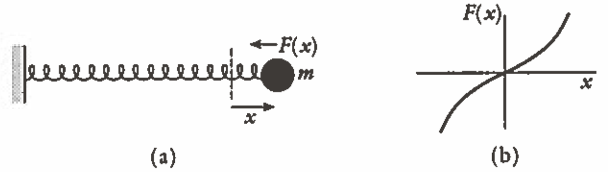

Fig. 5.2 (a) A spring oscillator. (b) A typical graph of spring force F(x) as a function of displacement x from equilibrium.

general this will be a complicated function of x, determined by the detailed elastic properties of the spring, but we do know that $F(0)=0$, because the spring force must be zero at the equilibrium point $x=0$.

Now, the exact equation of motion is

$$m\frac{d^2x}{dt^2} = -F(x), \tag{5.7}$$

but if we confine attention to *small* values of $|x|$, so that the particle is close to the equilibrium point, we may approximate $F(x)$ by the first two terms of its Taylor series about $x=0$:

$$F(x) \doteqdot F(0) + xF'(0) \tag{5.8}$$

(see (2.14)), and because $F(0)=0$ we may write

$$F(x) \doteqdot \alpha x, \tag{5.9}$$

where $\alpha = F'(0)$ denotes the positive 'spring constant.'

In this way, then, we again obtain a *linearized* equation of motion:

$$m\frac{d^2x}{dt^2} = -\alpha x, \tag{5.10}$$

approximately valid if $|x|$ is small. On writing

$$\omega^2 = \frac{\alpha}{m} \tag{5.11}$$

this becomes

$$\frac{d^2x}{dt^2} + \omega^2 x = 0, \tag{5.12}$$

with general solution

$$x = A\cos\omega t + B\sin\omega t \tag{5.13a}$$

(see (3.28)). This can also be written in the form

$$x = C\cos(\omega t - D), \tag{5.13b}$$

where $C=(A^2+B^2)^{1/2}$ and $D=\tan^{-1}(B/A)$.

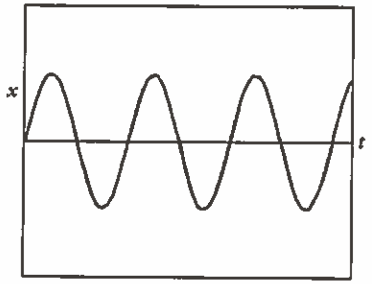

Fig. 5.3 A simple-harmonic oscillation.

Small oscillations about the equilibrium point are therefore simple harmonic (Fig. 5.3). The **period** of the oscillation is clearly $2\pi/\omega$, and the **frequency**, i.e. the number of oscillation cycles per unit time, is therefore $f=\omega/2\pi$. Having said this, we shall occasionally lapse into using the term 'frequency' to refer to ω itself; its proper title is *angular frequency*.

The effect of damping

Suppose now that the mass in Fig. 5.2 experiences also a frictional resistance which is proportional to its speed. This implies an additional force of $-\gamma\dot{x}$ in the positive x-direction, γ being a positive constant, and so

$$m\ddot{x} = -\alpha x - \gamma\dot{x}$$

is the new (approximate) equation of motion for small displacements x. On defining $k=\gamma/m$ we may rewrite this as

$$\ddot{x} + k\dot{x} + \omega^2 x = 0, \tag{5.14}$$

where ω is defined, as before, by (5.11).

It may be shown that the general solution to this equation is

$$x = Ce^{-kt/2}\cos\left\{\left(\omega^2 - \tfrac{1}{4}k^2\right)^{1/2}t - D\right\}, \tag{5.15}$$

provided that the damping is not too large, i.e. provided that $\frac{1}{4}k^2 < \omega^2$ (Ex. 5.1). The amplitude of the oscillations now decreases with time, in proportion to $e^{-kt/2}$, and the larger the frictional constant k the faster the decay of the oscillations (Fig. 5.4). A secondary effect of the damping is to reduce the oscillation frequency from ω to $(\omega^2 - \frac{1}{4}k^2)^{1/2}$.

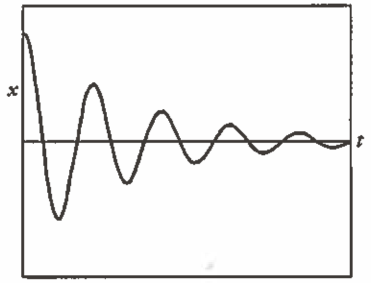

Fig. 5.4 An example of damped simple-harmonic motion.

Forced oscillations; resonance

Suppose finally that the particle in Fig. 5.2 is subject not only to the spring force $-\alpha x$ but to a prescribed *driving* force $F_0\cos\Omega t$, which is itself oscillatory. In the absence of damping the equation of motion for small $|x|$ is then

$$\ddot{x} + \omega^2 x = a\cos\Omega t, \tag{5.16}$$

where $a = F_0/m$.

Fig. 5.5 The forced linear spring.

This equation was studied by Euler in 1739, and he noted that the most interesting case is when the forcing frequency Ω coincides with the natural frequency of the spring ω. In this case we have the phenomenon of **resonance**, resulting in steadily growing oscillations. When $\Omega = \omega$ we may verify, for example, that

$$x = \frac{a}{2\omega} t \sin \omega t \tag{5.17}$$

satisfies (5.16) and the initial conditions $x = \dot{x} = 0$ at $t = 0$ (Ex. 5.2), so that the amplitude of the oscillation increases in proportion to t as time goes on.

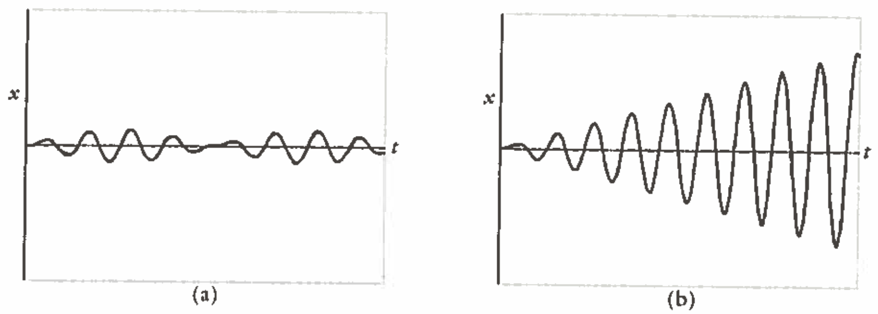

Fig. 5.6 Typical solutions to (5.16) with (a) $\Omega = 0.8\omega$, (b) $\Omega = \omega$.

If a little damping is present, the amplitude begins by growing in this way, but eventually settles down to a (relatively large) constant value (Fig. 5.7(a)). This final steady state response is largest when Ω is close to the resonant value ω (Fig. 5.7(b)).

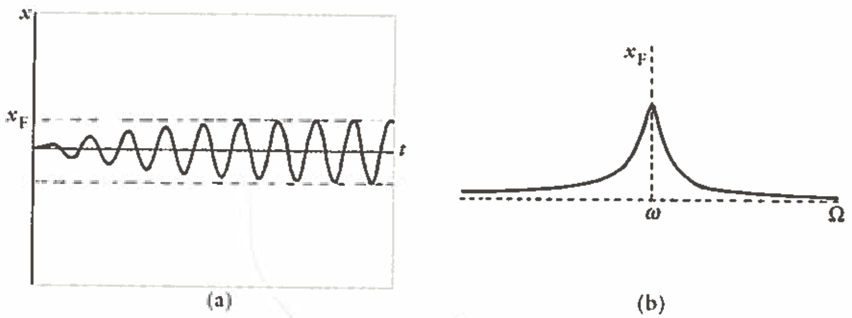

Fig. 5.7 The effect of small damping on resonant oscillations.

In practice, resonant oscillations may well be large enough that the linearized equation (5.16) ceases to be valid, because the approximation (5.9) breaks down.

5.3 Multiple modes of oscillation

Suppose now that two particles of mass m are attached to three identical springs of natural length a and constrained to move along a straight line, as in Fig. 5.8. We now need *two* coordinates to describe the system at any time t, and the system is said to have two **degrees of freedom**. If x_1 and x_2 denote small displacements from equilibrium, and if each spring has 'spring constant' α, then the tensions in the springs are

$$T_1 = \alpha x_1, \qquad T_2 = \alpha(x_2 - x_1), \qquad T_3 = -\alpha x_2, \tag{5.18}$$

because x_1, $x_2 - x_1$ and $-x_2$ are the amounts by which each spring is extended.

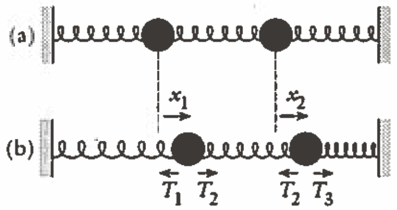

Fig. 5.8 An oscillating system with two degrees of freedom.

At time t, therefore, the first mass experiences a force $T_2 - T_1 = \alpha(x_2 - 2x_1)$, while the second experiences a force $T_3 - T_2 = \alpha(x_1 - 2x_2)$. The equations of motion are then

$$\begin{aligned} m\ddot{x}_1 &= \alpha(x_2 - 2x_1), \\ m\ddot{x}_2 &= \alpha(x_1 - 2x_2), \end{aligned} \tag{5.19a, b}$$

which are simultaneous differential equations for the two unknowns $x_1(t)$ and $x_2(t)$.

An obvious question which comes to mind is whether this system has a natural frequency of oscillation, as in the case of one degree of freedom (see (5.11) and Fig. 5.2(a)). We try, therefore,

$$x_1 = A \cos \omega t, \qquad x_2 = B \cos \omega t \tag{5.20}$$

and find that this does indeed represent a solution of (5.19a, b) if the constants A, B and ω are such that

$$\begin{aligned} -m\omega^2 A &= \alpha(B - 2A), \\ -m\omega^2 B &= \alpha(A - 2B). \end{aligned}$$