LOW DIMENSIONAL MILNOR-WITT STEMS OVER $\mathbb{R}$

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Abstract. This article computes some motivic stable homotopy groups over $\mathbb{R}$. For $0 \leq p - q \leq 3$, we describe the motivic stable homotopy groups $\hat{\pi}_{p,q}$ of a completion of the motivic sphere spectrum. These are the first four Milnor-Witt stems. We start with the known Ext groups over $\mathbb{C}$ and apply the ρ-Bockstein spectral sequence to obtain Ext groups over $\mathbb{R}$. This is the input to an Adams spectral sequence, which collapses in our low-dimensional range.

1. Introduction

This paper takes place in the context of motivic stable homotopy theory over $\mathbb{R}$. Write $\mathbb{M}_2 = H^{*,*}(\mathbb{R}; \mathbb{F}_2)$ for the bigraded motivic cohomology ring of a point, and write $\mathcal{A}$ for the motivic Steenrod algebra at the prime 2. Our goal is to study the tri-graded Adams spectral sequence

$$E_2 = \mathrm{Ext}_{\mathcal{A}}^{*,*,*}(\mathbb{M}_2, \mathbb{M}_2) \Rightarrow \hat{\pi}_{*,*},$$

where $\hat{\pi}_{*,*}$ represents the stable motivic homotopy groups of a completion of the motivic sphere spectrum over $\mathbb{R}$. Specifically, in a range of dimensions we

(1) Compute the Ext groups appearing in the E_2-page of the motivic Adams spectral sequence over $\mathbb{R}$.
(2) Analyze all Adams differentials.
(3) Reconstruct the groups $\hat{\pi}_{*,*}$ from their filtration quotients given by the Adams E_∞-page.

Point (1) is tackled by introducing an auxilliary, purely algebraic spectral sequence that converges to these Ext groups.

To describe our results more specifically we must introduce some notation and terminology related to the three indices in our spectral sequence. We first have the homological degree of the Ext groups, also called the Adams filtration degree—we label this f and simply call it the *filtration*. We then have the internal bidegree (t, w) for $\mathcal{A}$-modules, where t is the usual *topological degree* and w is the *weight*. We introduce the grading $s = t - f$ and call this the *topological stem*, or just the *stem*. The triple (s, f, w) of stem, filtration, and weight will be our main index of reference. Using these variables, the motivic Adams spectral sequence can be written

$$E_2 = \mathrm{Ext}_{\mathcal{A}}^{s,f,w}(\mathbb{M}_2, \mathbb{M}_2) \Rightarrow \hat{\pi}_{s,w}.$$

Morel has computed [15] that $\hat{\pi}_{s,w} = 0$ for $s < w$ (in fact, this is true integrally before completion). Write $\Pi_0 = \oplus_n \pi_{n,n}$, considered as a $\mathbb{Z}$-graded ring. This is called the *Milnor-Witt ring*, and Morel has given a complete description of this

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via generators and relations [14]. It is convenent to set $\Pi_k = \oplus_n \hat{\pi}_{n+k,n}$, as this is a $\mathbb{Z}$-graded module over Π_0. We call Π_k the completed **Milnor-Witt k-stem**. Related to this, the group $\hat{\pi}_{s,w}$ has *Milnor-Witt degree* $s-w$.

The completed Milnor-Witt ring Π_0 is equal to

$$\mathbb{Z}_2[\rho,\eta]/(\eta^2\rho + 2\eta, \rho^2\eta + 2\rho),$$

where η has degree $(1,1)$ and ρ has degree $(-1,-1)$. Note that Π_0 is the 2-completion of the Milnor-Witt ring of $\mathbb{R}$ described by Morel [14].

We have found that the analysis of the motivic Ext groups over $\mathbb{R}$, and of the Adams spectral sequence, is most conveniently done with respect to the Milnor-Witt degree. In this paper we focus only on the range $s-w \leq 3$, leading to an analysis of the Milnor-Witt stems Π_1, Π_2, and Π_3. The restriction to $s-w \leq 3$ is done for didactic purposes; our methods can be applied to cover a much greater range, but at the expense of more laborious computation. The focus on $s-w \leq 3$ allows us to demonstrate the methods and see examples of the interesting phenomena, while keeping the intensity of the labor down to manageable levels.

1.1. **An algebraic spectral sequence for Ext.** The main tool in this paper is the ρ-Bockstein spectral sequence that computes the groups $\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2,\mathbb{M}_2)$. This was originally introduced by Hill [6] and analyzed for the subalgebra $\mathcal{A}(1)$ of $\mathcal{A}$ generated by Sq^1 and Sq^2. Most of our hard work is focused on analyzing the differentials in this spectral sequence, as well as the hidden extensions encountered when passing from the E_∞-page to the true Ext groups.

Over the ground field $\mathbb{R}$, one has $\mathbb{M}_2 = \mathbb{F}_2[\tau,\rho]$ where τ has bidegree $(0,1)$ and ρ has bidegree $(1,1)$. In contrast, over $\mathbb{C}$ one has $\mathbb{M}_2^{\mathbb{C}} = \mathbb{F}_2[\tau]$. Write $\mathcal{A}^{\mathbb{C}}$ for the motivic Steenrod algebra over $\mathbb{C}$. The groups $\mathrm{Ext}^{s,f,w}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}},\mathbb{M}_2^{\mathbb{C}})$ were computed for $s \leq 34$ in [3], and then for $s \leq 70$ in [9]. The ρ-Bockstein spectral sequence takes these groups as input, having the form

$$E_1 = \mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}},\mathbb{M}_2^{\mathbb{C}})[\rho] \Rightarrow \mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2,\mathbb{M}_2). \tag{1.2}$$

The differentials in this spectral sequence are extensive. However, in a large range they can be completely analyzed by a method we describe next.

As an $\mathbb{F}_2[\rho]$-module, $\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2,\mathbb{M}_2)$ splits as a summand of ρ-torsion modules and ρ-non-torsion modules; we call the latter *ρ-local* modules for short. The first step in our work is to analyze the ρ-local part of the Ext groups, and this turns out to have a remarkably simple answer. We prove that

$$\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2,\mathbb{M}_2)[\rho^{-1}] \cong \mathrm{Ext}_{\mathcal{A}_{cl}}(\mathbb{F}_2,\mathbb{F}_2)[\rho^{\pm 1}]$$

where $\mathcal{A}_{cl}$ is the classical Steenrod algebra at the prime 2. The isomorphism is highly structured, in the sense that it is compatible with all products and Massey products, and the element h_i in $\mathrm{Ext}_{\mathcal{A}_{cl}}(\mathbb{F}_2,\mathbb{F}_2)$ corresponds to the element h_{i+1} in $\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2,\mathbb{M}_2)[\rho^{-1}]$ for every $i \geq 0$. In other words, the motivic Ext groups $\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2,\mathbb{M}_2)$ have a shifted copy of $\mathrm{Ext}_{\mathcal{A}_{\mathrm{cl}}}(\mathbb{F}_2,\mathbb{F}_2)$ sitting inside them as the ρ-local part.

It turns out that through a large range of dimensions, there is only one pattern of ρ-Bockstein differentials that is consistent with the ρ-local calculation described in the previous paragraph. This is what allows the analysis of the ρ-Bockstein spectral sequence (1.2). It is not so easy to organize this calculation: the tri-graded nature of the spectral sequence, coupled with a fairly irregular pattern of differentials, makes it close to impossible to depict the spectral sequence via the usual charts. We analyze what is happening via a collection of charts and tables, but mostly focusing on the tables. A large portion of the present paper is devoted to explaining how to navigate this computation.

Figure 3 shows the result (all figures are contained in Section 10). This figure displays $\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)$ through Milnor-Witt degree 4. Our computations agree with machine computations carried out by Glen Wilson and Knight Fu [19].

1.3. **Adams differentials.** Once we have computed $\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)$, the next step is the analysis of Adams differentials. Identifying even *possible* differentials is again hampered by the tri-graded nature of the situation, but we explain the calculus that allows one to accomplish this—it is not as easy as it is for the classical Adams spectral sequence, but it is at least mechanical. In the range $s - w \leq 3$ there are only a few possible differentials for degree reasons. We show via some Toda bracket arguments that in fact all of the differentials are zero.

1.4. **Milnor-Witt modules.** After analyzing Adams differentials, we obtain the Adams E_∞-page, which is an associated graded object of the motivic stable homotopy groups over $\mathbb{R}$. We convert the associated graded information into the structure of the Milnor-Witt modules Π_1, Π_2, and Π_3, as modules over Π_0. We must be wary of extensions that are hidden by the Adams spectral sequence, but these turn out to be manageable.

Figure 4 describes the results of this process. We draw attention to a curious phenomenon in the 7-stem of Π_3. Here we see that the third Hopf map σ has order 32, not order 16. This indicates that the motivic image of J is not the same as the classical image of J. This unexpected behavior suggests that the theory of motivic (and perhaps equivariant) v_1-self maps is not what one might expect. These phenomena deserve more study.

We also observe that the 1-stem of Π_1 is consistent with Morel's conjecture on the structure of $\pi_{1,0}$. (See [17, p. 98] for a clearly stated version of the conjecture.)

Unsurprisingly, our calculations are similar to calculations of $\mathbb{Z}/2$-equivariant stable homotopy groups [1]. There is a realization functor from motivic homotopy theory over $\mathbb{R}$ to $\mathbb{Z}/2$-equivariant homotopy theory, and this functor induces an isomorphism in stable homotopy groups in a range. We will return to this comparison in future work.

1.5. **Other base fields.** Although we only work with the base field $\mathbb{R}$ in this article, the phenomena that we study most likely occur for other base fields as well. This is especially true for fields k that are similar to $\mathbb{R}$, such as fields that have an embedding into $\mathbb{R}$.

One might use our calculations to speculate on the structure of Π_1, Π_2, and Π_3 for arbitrary base fields. We leave this to the imagination of the reader.

1.6. **Organization of the paper.** We begin in Section 2 with a brief reminder of the motivic Steenrod algebra and the motivic Adams spectral sequence. We construct the ρ-Bockstein spectral sequence in Section 3, and we perform some preliminary calculations. In Section 4, we consider the effect of inverting ρ. Then we return in Section 5 to a detailed analysis of the ρ-Bockstein spectral sequence. We resolve extensions that are hidden in the ρ-Bockstein spectral sequence in Section 6, and obtain a description of $\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)$. We show that there are no Adams differentials in Section 7. In Section 8, we convert the associated graded information of the Adams spectral sequence into explicit descriptions of Milnor-Witt modules. Sections 9 and 10 contain the tables and charts required to carry out our detailed computations. We have collected this essential information in one place for the convenience of readers who are seeking specific computational facts.

1.7. **Notation.** For the reader's convenience, we provide a table of notation to be used later.

(1) $\mathbb{M}_2 = \mathbb{F}_2[\tau, \rho]$ is the motivic $\mathbb{F}_2$-cohomology ring of $\mathbb{R}$.
(2) $\mathbb{M}_2^{\mathbb{C}} = \mathbb{F}_2[\tau$ is the motivic $\mathbb{F}_2$-cohomology ring of $\mathbb{C}$.
(3) $\mathcal{A}$ is the motivic Steenrod algebra over $\mathbb{R}$ at the prime 2.
(4) $\mathcal{A}_*$ is the dual motivic Steenrod alegbra over $\mathbb{R}$ at the prime 2.
(5) $\mathcal{A}^{\mathbb{C}}$ is the motivic Steenrod algebra over $\mathbb{C}$ at the prime 2.
(6) $\mathcal{A}_{\mathrm{cl}}$ is the classical Steenrod algebra at the prime 2.
(7) Ext or $\mathrm{Ext}_{\mathbb{R}}$ is the cohomology of $\mathcal{A}$, i.e., $\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)$.
(8) $\mathrm{Ext}_{\mathbb{C}}$ is the cohomology of $\mathcal{A}^{\mathbb{C}}$, i.e., $\mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})$.
(9) $\mathrm{Ext}_{\mathrm{cl}}$ is the cohomology of $\mathcal{A}^{\mathrm{cl}}$, i.e., $\mathrm{Ext}_{\mathcal{A}^{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$.
(10) $\hat{\pi}_{*,*}$ is the bigraded stable homotopy ring of the completion of the motivic sphere spectrum over $\mathbb{R}$ with respect to the motivic Eilenberg-Mac Lane spectrum $H\mathbb{F}_2$.
(11) $\Pi_k = \oplus_n \hat{\pi}_{n+k,k}$ is the kth completed Milnor-Witt stem over $\mathbb{R}$.

2. Background

This section establishes the basic setting and notation that will be assumed throughout the paper.

Write $\mathbb{M}_2 = H^{*,*}(\mathbb{R}; \mathbb{F}_2)$ for the (bigraded) motivic cohomology ring of $\mathbb{R}$. We use the usual motivic bigrading where the first index is the topological dimension and the second index is the weight. Recall that $\mathbb{M}_2$ is equal to $\mathbb{F}_2[\tau, \rho]$, where τ has degree $(0, 1)$ and ρ has degree $(1, 1)$. The class ρ is the element $[-1]$ under the standard isomorphism $\mathbb{M}_2^{1,1} \cong F^*/(F^*)^2$, and τ is the unique element such that $\mathrm{Sq}^1(\tau) = \rho$.

Let $\mathcal{A}_*$ denote the dual motivic Steenrod algebra over $\mathbb{R}$. The pair $(\mathbb{M}_2, \mathcal{A}_*)$ is a Hopf algebroid; recall from [20] (see also [2]) that this structure is described by

$$\mathcal{A}_* = \mathbb{M}_2[\tau_0, \tau_1, \ldots, \xi_0, \xi_1, \ldots]/(\xi_0 = 1, \tau_k^2 = \tau \xi_{k+1} + \rho \tau_{k+1} + \rho \tau_0 \xi_{k+1})$$
$$\eta_L(\tau) = \tau, \quad \eta_R(\tau) = \tau + \rho \tau_0, \quad \eta_L(\rho) = \eta_R(\rho) = \rho$$
$$\Delta(\tau_k) = \tau_k \otimes 1 + \sum \xi_{k-i}^{2^i} \otimes \tau_i$$
$$\Delta(\xi_k) = \sum \xi_{k-i}^{2^i} \otimes \xi_i.$$

The Hopf algebroid axioms force $\Delta(\tau) = \tau \otimes 1$ and $\Delta(\rho) = \rho \otimes 1$, but it is useful to record these for reference. The dual $\mathcal{A}_*$ is homologically graded, so τ has degree $(0, -1)$ and ρ has degree $(-1, -1)$. Moreover, τ_k has degree $(2^{i+1} - 1, 2^i - 1)$ and ξ_k has degree $(2^{i+1} - 2, 2^i - 1)$.

The groups $\mathrm{Ext}_{\mathcal{A}_*}(\mathbb{M}_2, \mathbb{M}_2)$ are trigraded. There is the homological degree f (the degree on the Ext) and the internal bidegree (p, q) of $\mathcal{A}_*$-comodules. The symbol f comes from 'filtration', as this index coincides with the Adams filtration in the Adams spectral sequence. Classical notation would write $\mathrm{Ext}^{f,(p,q)}$ for the corresponding homogeneous piece of the Ext group. In the Adams spectral sequence this Ext group contribues to $\pi_{p-f,q}$. We call $p-f$ the **stem** and will usually denote it by s. It turns out to be more convenient to use the indices (s,f,w) of stem, filtration, and weight rather than (f,p,q). So we will write $\mathrm{Ext}^{s,f,w}$ for the group that would classically be denoted $\mathrm{Ext}^{f,(s+f,w)}$. This works very well in practice; in particular, when we draw charts, the group $\mathrm{Ext}^{s,f,w}$ will be located at Cartesian coordinates (s,f).

The motivic Adams spectral sequence takes the form

$$E_2 = \mathrm{Ext}^{s,f,w}_{\mathcal{A}_*}(\mathbb{M}_2, \mathbb{M}_2) \Rightarrow \hat{\pi}_{s,w},$$

with $d_r\colon \mathrm{Ext}^{s,f,w} \to \mathrm{Ext}^{s-1,f+r,w}$. Here $\hat{\pi}_{*,*}$ is the stable motivic homotopy ring of the completion of the motivic sphere spectrum with respect to the motivic Eilenberg-Mac Lane spectrum $H\mathbb{F}_2$. (According to [8], this completion is also the 2-completion of the motivic sphere spectrum, but this is not essential for our calculations.)

Our methods also require us to consider the motivic cohomology of $\mathbb{C}$ and the motivic Steenrod algebra over $\mathbb{C}$. We write $\mathbb{M}_2^{\mathbb{C}}$ and $\mathcal{A}^{\mathbb{C}}$ for these objects. They are obtained from $\mathbb{M}_2$ and $\mathcal{A}$ by setting ρ equal to zero. More explicitly, $\mathbb{M}_2^{\mathbb{C}}$ equals $\mathbb{F}_2[\tau]$, and the dual motivic Steenrod alegbra over $\mathbb{C}$ has relations of the form $\tau_k^2 = \tau\xi_{k+1}$.

We will also abbreviate

$$\mathrm{Ext}_{\mathbb{R}} = \mathrm{Ext}_{\mathcal{A}_*}(\mathbb{M}_2, \mathbb{M}_2), \qquad \mathrm{Ext}_{\mathbb{C}} = \mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}_*}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}}).$$

2.1. **Milnor-Witt degree.** Given a class with an associated stem s and weight w, we call $s-w$ the **Milnor-Witt degree** of the class. The terminology comes from the fact that the elements of Milnor-Witt degree zero in the motivic stable homotopy ring constitute Morel's Milnor-Witt K-theory ring. More generally, the elements of Milnor-Witt degree r in $\hat{\pi}_{*,*}$ form a module over (2-completed) Milnor-Witt K-theory.

Many of the calculations in this paper are handled by breaking things up into the homogeneous Milnor-Witt components. The following lemma about $\mathrm{Ext}_{\mathbb{C}}$ will be particularly useful.

Lemma 2.2. *Let x be a non-zero class in $\mathrm{Ext}^{s,f,w}_{\mathbb{C}}$ with Milnor-Witt degree t. Then $f \geq s-2t$.*

Proof. The motivic May spectral sequence [3] has E_1-page generated by classes h_{ij}, and converges to $\mathrm{Ext}^{\mathbb{C}}$. All of the classes h_{ij} are readily checked to satisfy the inequality $s+f-2w \geq 0$, and this extends to all products.

This inequality is the same as $f \geq s-2(s-w)$, and t equals $s-w$ by definition. □

In practice, Lemma 2.2 tells us where to look for elements of $\mathrm{Ext}_{\mathbb{C}}$ in a given Milnor-Witt degree. All such elements lie above a line of slope 1 on an Adams chart.

3. The ρ-Bockstein spectral sequence

Our aim is to compute $\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)$. What makes this calculation difficult is the presence of ρ. If one formally sets $\rho = 0$ then the formulas become simpler and the calculations more manageable; this is essentially the case that was handled in

[3] and [9]. Following ideas of Hill [6], we use an algebraic spectral sequence for building up the general calculation from the simpler one where $\rho = 0$. This section sets up the spectral sequence and establishes some basic properties.

Let $\mathcal{C}$ be the (unreduced) cobar complex for the Hopf algebroid $(\mathbb{M}_2, \mathcal{A}_*)$. Recall that this is the cochain complex associated to the cosimplicial ring

$$\mathbb{M}_2 \underset{\eta_L}{\overset{\eta_R}{\rightrightarrows}} \mathcal{A}_* \Rrightarrow \mathcal{A}_* \otimes_{\mathbb{M}_2} \mathcal{A}_* \Rrightarrow \mathcal{A}_* \otimes_{\mathbb{M}_2} \mathcal{A}_* \otimes_{\mathbb{M}_2} \mathcal{A}_* \cdots$$

by taking $d_{\mathcal{C}}$ to be the alternating sum of the coface maps. For u an r-fold tensor, one has $d^0(u) = 1 \otimes u$, $d^{r+1}(u) = u \otimes 1$, and $d^i(u)$ applies the diagonal of $\mathcal{A}_*$ to the ith tensor factor of u. For u in $\mathbb{M}_2$ (i.e., a 0-fold tensor), one has $d^0(u) = \eta_R(u)$ and $d^1(u) = \eta_L(u)$.

The pair $(\mathcal{C}, d_{\mathcal{C}})$ is a differential graded algebra. As usual, we will denote r-fold tensors via the bar notation $[x_1|x_2|\cdots|x_r]$.

The element $\xi_1^{2^k}$ is primitive in $\mathcal{A}_*$ for any k because ξ_1 is primitive. Hence $[\xi_1^{2^k}]$ is a cycle in the cobar complex that is denoted h_{k+1}. Likewise, τ_0 is primitive, and the cycle $[\tau_0]$ is denoted h_0.

The maps η_L, η_R, and Δ all fix ρ, and this implies that all the coface maps are ρ-linear. The filtration

$$\mathcal{C} \supseteq \rho\mathcal{C} \supseteq \rho^2\mathcal{C} \supseteq \cdots$$

is therefore a filtration of chain complexes. The associated spectral sequence is called the **ρ-Bockstein spectral sequence**.

The ρ-Bockstein spectral sequence has the form

$$E_1 = \operatorname{Ext}_{\operatorname{Gr}_\rho \mathcal{A}}(\operatorname{Gr}_\rho \mathbb{M}_2, \operatorname{Gr}_\rho \mathbb{M}_2) \Rightarrow \operatorname{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2),$$

where Gr_ρ refers to the associated graded of the filtration by powers of ρ. Since $\mathbb{M}_2 = \mathbb{F}_2[\tau, \rho]$, we have $\operatorname{Gr}_\rho \mathbb{M}_2 \cong \mathbb{M}_2$. Similarly, it follows easily that there is an isomorphism of Hopf algebroids

$$(\operatorname{Gr}_\rho \mathbb{M}_2, \operatorname{Gr}_\rho \mathcal{A}) \cong (\mathbb{M}_2^{\mathbb{C}}, \mathcal{A}^{\mathbb{C}}) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\rho],$$

where $\mathbb{M}_2^{\mathbb{C}} = \mathbb{F}_2[\tau]$ is the motivic cohomology ring of $\mathbb{C}$. The point here is that after taking associated gradeds, the formulas for η_L and η_R both fix τ, whereas the formulas for Δ are unchanged; and all of this exactly matches the formulas for $\mathcal{A}^{\mathbb{C}}$. Tensoring with $\mathbb{F}_2[\rho]$ commutes with Ext, and so our ρ-Bockstein spectral sequence takes the form

$$E_1 = \operatorname{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})[\rho] \Rightarrow \operatorname{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2).$$

It will be convenient to denote $\operatorname{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})$ simply by $\operatorname{Ext}_{\mathbb{C}}$.

We observe two general properties of the ρ-Bockstein spectral sequence. First, the element ρ is a permanent cycle because ρ supports no Steenrod operations. Second, the spectral sequence is multiplicative, so the Leibniz rule can be used effectively to compute differentials on decomposable elements.

Remark 3.1. Here is a method for deducing ρ-Bockstein differentials from explicit cobar calculations. Let u be an element in $\mathcal{C}$, and assume that u is not a multiple of ρ. If possible, write $d_{\mathcal{C}}(u) = \rho d_{\mathcal{C}}(u_1) + \rho^2 v_2$, where u_1 has a tensor expression that does not involve ρ; then the ρ-Bockstein differential $d_1(u)$ is zero. Otherwise, $d_1(u)$ equals $d_{\mathcal{C}}(u)$ modulo ρ^2.

If $d_1(u)$ is zero, then if possible write $d_{\mathcal{C}}(u) = \rho d_{\mathcal{C}}(u_1) + \rho^2 d_{\mathcal{C}}(u_2) + \rho^3 v_3$, where u_2 has a tensor expression that does not involve ρ; then $d_2(u)$ is zero. Otherwise, $d_2(u)$ equals $\rho^2 v_2$ modulo ρ^3.

Inductively, assume that

$$d_{\mathcal{C}}(u) = \rho d_{\mathcal{C}}(u_1) + \cdots + \rho^{r-1} d_{\mathcal{C}}(u_{r-1}) + \rho^r v_r,$$

where each u_i has a tensor expression that does not involve ρ. If possible, write $v_r = d_{\mathcal{C}}(u_r) + \rho v_{r+1}$, where u_r has a tensor expression that does not involve ρ; then $d_r(u)$ is zero. Otherwise, $d_r(u)$ equals $\rho^r v_r$ modulo ρ^{r+1}.

The method described in Remark 3.1 is mostly not needed in our analysis; in fact, we will eventually show how to deduce a most of the differentials in a large range of the spectral sequence by a completely mechanical process. Still, it is often useful to understand that the ρ-Bockstein spectral sequence is all about computing ρ-truncations of differentials in $\mathcal{C}$. Proposition 3.2 and Example 3.3 illustrate this technique.

Proposition 3.2.

(1) $d_1(\tau) = \rho h_0$.
(2) $d_{2^k}(\tau^{2^k}) = \rho^{2^k} \tau^{2^{k-1}} h_k$ *for* $k \geq 1$.

Part (2) of Proposition 3.2 implicitly also means that $d_r(\tau^{2^k})$ is zero for all $r < 2^k$.

Proof. Note that $d_{\mathcal{C}}(x) = \eta_R(x) - \eta_L(x)$ for x in $\mathbb{M}_2$. In particular, $d_{\mathcal{C}}(\tau) = [\tau + \rho\tau_0] - [\tau] = \rho[\tau_0] = \rho h_0$. Now use Remark 3.1 to deduce that $d_1(\tau) = \rho h_0$.

Next we analyze $d_{\mathcal{C}}(\tau^{2^k})$. Start with

$$d_{\mathcal{C}}(\tau^{2^k}) = \eta_R(\tau^{2^k}) - \eta_L(\tau^{2^k}) = [(\tau + \rho\tau_0)^{2^k}] - [\tau^{2^k}] = \rho^{2^k}[\tau_0^{2^k}].$$

Recall that $\tau_0^2 = \tau\xi_1 + \rho\tau_1 + \rho\tau_0\xi_1$ in $\mathcal{A}_*$, and so $\tau_0^{2^k} = \tau^{2^{k-1}} \xi_1^{2^{k-1}}$ modulo $\rho^{2^{k-1}}$. Thus, $d_{\mathcal{C}}(\tau^{2^k}) = \rho^{2^k} \tau^{2^{k-1}} [\xi_1^{2^{k-1}}]$ modulo ρ^{2^k+1}. Remark 3.1 implies that $d_{2^k}(\tau^{2^k}) = \rho^{2^k} \tau^{2^{k-1}} h_k$. □

Example 3.3. We will demonstrate that $d_6(\tau^4 h_1) = \rho^6 \tau h_2^2$. As in the proof of Proposition 3.2, $d_{\mathcal{C}}(\tau^4) = \rho^4[(\tau\xi_1 + \rho\tau_1 + \rho\tau_0\xi_1)^2]$. Use the relations $\tau_0^2 = \tau\xi_1 + \rho\tau_1 + \rho\tau_0\xi_1$ and $\tau_1^2 = \tau\xi_2 + \rho\tau_2 + \rho\tau_0\xi_2$ to see that this expression equals $\rho^4\tau^2[\xi_1^2] + \rho^6\tau[\xi_2] + \rho^6\tau[\xi_1^3]$ modulo ρ^7.

Since $h_1 = [\xi_1]$ is a cycle, we therefore have

$$d_{\mathcal{C}}(\tau^4 h_1) = \rho^4\tau^2[\xi_1^2|\xi_1] + \rho^6\tau\big([\xi_2|\xi_1] + [\xi_1^3|\xi_1]\big)$$

modulo ρ^7.

The coproduct on ξ_2 implies that $d_{\mathcal{C}}([\xi_2]) = [\xi_1^2|\xi_1]$. We also have that

$$d_{\mathcal{C}}(\tau^2) = \rho^2[\tau_0^2] = \rho^2\tau[\xi_1] + \rho^3[\tau_1] + \rho^2[\tau_0\xi_1],$$

as in the proof of Proposition 3.2. Therefore, the Leibniz rule gives that

$$d_{\mathcal{C}}(\tau^2[\xi_2]) = \rho^2\tau[\xi_1|\xi_2] + \rho^3[\tau_1|\xi_2] + \rho^3[\tau_0\xi_1|\xi_2] + \tau^2[\xi_1^2|\xi_1].$$

We can now write

$$d_{\mathcal{C}}(\tau^4 h_1) = \rho^4 d_{\mathcal{C}}(\tau^2[\xi_2]) + \rho^6\tau\big([\xi_2|\xi_1] + [\xi_1^3|\xi_1] + [\xi_1|\xi_2]\big)$$

modulo ρ^7. From Remark 3.1, one has $d_i(\tau^4 h_1) = 0$ for $i < 6$ in the ρ-Bockstein spectral sequence, and $d_6(\tau^4 h_1) = \rho^6\tau\big([\xi_2|\xi_1] + [\xi_1^3|\xi_1] + [\xi_1|\xi_2]\big)$.

Finally, the coproduct in $\mathcal{A}_*$ implies that

$$d_{\mathcal{C}}([\xi_2\xi_1]) = [\xi_1^3|\xi_1] + [\xi_1|\xi_2] + [\xi_2|\xi_1] + [\xi_1^2|\xi_1^2].$$

This shows that $[\xi_2|\xi_1] + [\xi_1^3|\xi_1] + [\xi_1|\xi_2] = h_2^2$ in Ext.

The long analysis in Example 3.3 demonstrates that direct work with the cobar complex is not practical. Instead, we will use some clever tricks that take advantage of various algebraic structures. But it is useful to remember what is going on behind the scenes: these computations of differentials are always giving us clues about the cobar differential $d_{\mathcal{C}}$.

The following two results are useful in analyzing ρ-Bockstein differentials.

Lemma 3.4. *If $d_r(x)$ is non-trivial in the ρ-Bockstein spectral sequence, then x and $d_r(x)$ are both ρ-torsion free on the E_r-page.*

Proof. First note that if y is nonzero on the E_r-page, then y is ρ-torsion if and only if $\rho^{r-1}y = 0$. The reason is that the differentials d_s for $s < r$ can only hit ρ^s-multiples of y.

Now suppose that $d_r(x) = \rho^r y$, where $\rho^r y$ is non-zero on the E_r-page. This immediately forces y to be ρ-torsion free. Since d_r is ρ-linear, this implies that x must also be ρ-torsion free on the E_r-page. □

4. ρ-localization

The analysis of the ρ-Bockstein spectral sequence is best broken up into two pieces. There are a large number of ρ-torsion classes in the E_∞-page. If one throws away all of this ρ-torsion, then the end result turns out to be fairly simple. In this section we compute this simple piece of $\text{Ext}_{\mathbb{R}}$. More precisely, we will consider the ρ-localization $\text{Ext}_{\mathbb{R}}[\rho^{-1}]$ of $\text{Ext}^{\mathbb{R}}$. Inverting ρ annihilates all of the ρ-torsion.

Let $\mathcal{A}^{\text{cl}}$ denote the classical Steenrod algebra (at the prime 2), and write $\text{Ext}_{\text{cl}} = \text{Ext}_{\mathcal{A}^{\text{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$.

Theorem 4.1. *There is an isomorphism from $\text{Ext}_{\text{cl}}[\rho^{\pm 1}]$ to $\text{Ext}_{\mathbb{R}}[\rho^{-1}]$ such that:*

1. *The isomorphism is highly structured, i.e., preserves products, Massey products, and algebraic squaring operations in the sense of* [12].
2. *The element h_n of Ext_{cl} corresponds to the element h_{n+1} of $\text{Ext}_{\mathbb{R}}$.*
3. *An element in Ext_{cl} of degree (s, f) corresponds to an element in $\text{Ext}_{\mathbb{R}}$ of degree $(2s + f, f, s + f)$.*

The formula for degrees appears to be more complicated than it is. The idea is that one doubles the internal degree, which is the stem plus the Adams filtration, while leaving the Adams filtration unchanged. Then the weight is always exactly half of the internal degree.

Proof. Since localization is exact, we may compute the cohomology of the Hopf algebroid $(\mathbb{M}_2[\rho^{-1}], \mathcal{A}_*[\rho^{-1}])$ to obtain $\text{Ext}^{\mathbb{R}}[\rho^{-1}]$. After localizing at ρ, we have $\tau_{k+1} = \rho^{-1}\tau_k^2 + \rho^{-1}\tau\xi_{k+1} + \tau_0\xi_{k+1}$, and so the Hopf algebroid $(\mathbb{M}_2[\rho^{-1}], \mathcal{A}_*[\rho^{-1}])$

is described by

$$\begin{aligned}
&\mathcal{A}_*[\rho^{-1}] = \mathbb{M}_2[\rho^{-1}][\tau_0, \xi_0, \xi_1, \ldots]/(\xi_0 = 1) \\
&\eta_L(\tau) = \tau, \quad \eta_R(\tau) = \tau + \rho\tau_0, \quad \eta_L(\rho) = \eta_R(\rho) = \rho \\
&\Delta(\tau_0) = \tau_0 \otimes 1 + 1 \otimes \tau_0 \\
&\Delta(\xi_k) = \sum \xi_{k-i}^{2^i} \otimes \xi_i.
\end{aligned}$$

Since these formulas contain no interactions between τ_i's and ξ_j's, there is a splitting

$$(\mathbb{M}_2[\rho^{-1}], \mathcal{A}_*[\rho^{-1}]) \cong (\mathbb{M}_2[\rho^{-1}], \mathcal{A}'_*) \otimes_{\mathbb{F}_2} (\mathbb{F}_2, \mathcal{A}''_*),$$

where $(\mathbb{M}_2[\rho^{-1}], \mathcal{A}'_*)$ is the Hopf algebroid

$$\begin{aligned}
&\mathcal{A}'_* = \mathbb{M}_2[\rho^{-1}][\tau_0] \\
&\eta_L(\tau) = \tau, \quad \eta_R(\tau) = \tau + \rho\tau_0 \\
&\Delta(\tau_0) = \tau_0 \otimes 1 + 1 \otimes \tau_0
\end{aligned}$$

and $(\mathbb{F}_2, \mathcal{A}''_*)$ is the Hopf algebra

$$\begin{aligned}
&\mathcal{A}''_* = \mathbb{F}_2[\xi_0, \xi_1, \ldots]/(\xi_0 = 1) \\
&\Delta(\xi_k) = \sum \xi_{k-i}^{2^i} \otimes \xi_i.
\end{aligned}$$

Notice that $\mathcal{A}''_*$ is equal to the classical dual Steenrod algebra, and so its cohomology is $\mathrm{Ext}_{\mathrm{cl}}$ (with degrees suitably shifted). For $\mathcal{A}'_*$, we can perform the change of variables $x = \rho\tau_0$ since ρ is invertible, yielding

$$(\mathbb{M}_2[\rho^{-1}], \mathcal{A}'_*) \cong \mathbb{F}_2[\rho^{\pm 1}] \otimes_{\mathbb{F}_2} (\mathbb{F}_2[\tau], \mathcal{B}),$$

where $(\mathbb{F}_2[\tau], \mathcal{B})$ is the Hopf algebroid defined in Lemma 4.2 below. The lemma implies that the cohomology of $(\mathbb{M}_2[\rho^{-1}], \mathcal{A}'_*)$ is $\mathbb{F}_2[\rho^{\pm 1}]$, concentrated in homological degree zero. □

Lemma 4.2. *Let $R = \mathbb{F}_2[t]$ and let $\mathcal{B} = R[x]$, with Hopf algebroid structure on $(R, \mathcal{B})$ given by the formulas*

$$\eta_L(t) = t, \quad \eta_R(t) = t + x, \quad \Delta(x) = x \otimes 1 + 1 \otimes x$$

(the formula $\Delta(t) = t \otimes 1$ is forced by the axioms). Then the cohomology of $(R, \mathcal{B})$ is isomorphic to $\mathbb{F}_2$, concentrated in homological degree 0.

Proof. Let $\mathcal{C}_{\mathcal{B}}$ be the cobar complex of $(R, \mathcal{B})$, and filter by powers of x. More explicitly, let $F_i\mathcal{C}_{\mathcal{B}}$ be the subcomplex

$$0 \to x^i\mathcal{B} \to \sum_{p+q=i} x^p\mathcal{B} \otimes_R x^q\mathcal{B} \to \sum_{p+q+r=i} x^p\mathcal{B} \otimes_R x^q\mathcal{B} \otimes_R x^r\mathcal{B} \to \cdots$$

This is indeed a subcomplex, and the associated graded $\mathrm{Gr}_x\, \mathcal{C}_{\mathcal{B}}$ is the cobar complex for $(R, \mathrm{Gr}_x\, \mathcal{B})$. This pair is isomorphic to the Hopf algebra (no longer a Hopf algebroid) where $\eta_L(t) = \eta_R(t) = t$ and $\Delta(x) = x \otimes 1 + 1 \otimes x$. The associated cohomology is the infinite polynomial algebra $\mathbb{F}_2[t, h_0, h_1, h_2, \ldots]$, where $h_i = [x^{2^i}]$. One easy way to see this is to note that the dual of $\mathrm{Gr}_x\, \mathcal{B}$ is the exterior algebra $\mathbb{F}_2[t](e_0, e_1, e_2, \ldots)$, where e_i is dual to x^{2^i}.

Our filtered cobar complex gives rise to a multiplicative spectral sequence with E_1-page equal to $\mathbb{F}_2[t, h_0, h_1, \ldots]$ and converging to the cohomology of $(R, \mathcal{B})$. The classes h_i are all infinite cycles, since $[x^{2^i}]$ is indeed a cocycle in $\mathcal{C}_{\mathcal{B}}$. Essentially the same analysis as in Proposition 3.2 shows that $d_1(t) = h_0$. This shows that the E_2-page is $\mathbb{F}_2[t^2, h_1, h_2, \ldots]$. The analysis from Proposition 3.2 again shows $d_2(t^2) = h_1$, which implies that the E_3-page is $\mathbb{F}_2[t^4, h_2, h_3, \ldots]$. Continue inductively, using that $d_{2^i}(t^{2^i}) = h_i$. The E_∞-page is just $\mathbb{F}_2$. □

Remark 4.3. We gave a calculational proof of Lemma 4.2. Here is a sketch of a more conceptual proof.

The Hopf algebroid $(R, \mathcal{B})$ has the same information as the presheaf of groupoids which sends an $\mathbb{F}_2$-algebra S to the groupoid with object set $\mathrm{Hom}_{\mathbb{F}_2\text{–alg}}(R, S)$ and morphism set $\mathrm{Hom}_{\mathbb{F}_2\text{–alg}}(\mathcal{B}, S)$. One readily checks that this groupoid is the translation category associated to the abelian group $(S, +)$; very briefly, the image of x in S is the name of the morphism, the image of t is its domain, and therefore $t + x$ is its codomain. Notice that this groupoid is contractible no matter what S is—this is the key observation. By [7, Theorems A and B] it follows that the category of $(R, \mathcal{B})$-comodules is equivalent to the category of comodules for the trivial Hopf algebroid $(\mathbb{F}_2, \mathbb{F}_2)$. In particular, one obtains an isomorphism of Ext groups.

5. Analysis of the ρ-Bockstein spectral sequence

In this section we determine all differentials in the ρ-Bockstein spectral sequence, within a given range of dimensions.

5.1. **Identification of the E_1-page.** From Section 3, the ρ-Bockstein spectral sequence takes the form

$$E_1 = \mathrm{Ext}_{\mathbb{C}}[\rho] \Rightarrow \mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2).$$

The groups $\mathrm{Ext}_{\mathbb{C}}$ have been computed in [3] and [9] through a large range of dimensions. Figure 1 gives a picture of $\mathrm{Ext}_{\mathbb{C}}$. Recall that this chart is a two-dimensional representation of a tri-graded object. For every black dot x in the chart there are classes $\tau^i x$ for $i \geq 1$ lying behind x (going into the page); in contrast, the red dots are killed by τ. To get the E_1-page for the ρ-Bockstein spectral sequence, we freely adjoin the class ρ to this chart. With respect to the picture, multiplication by ρ moves one degree to the left and one degree back. So we can regard the same chart as a depiction of our E_1-page if we interpret every black dot as representing an entire triangular cone moving back (via multiplication by τ) and to the left (via multiplication by ρ); and every red dot represents a line of ρ-multiples going back and to the left. For example, we must remember that in the $(2, 1)$ spot on the grid there are classes $\rho\tau^i h_2$, $\rho^5\tau^i h_3$, $\rho^{13}\tau^i h_4$, and so forth. In general, when looking at coordinates (s, f) on the chart, one must look horizontally to the right and be aware that $\rho^k x$ is potentially present, where x is a class in $\mathrm{Ext}_{\mathbb{C}}$ at coordinates $(s + k, f)$.

There are so many classes in the E_1-page, and it is so difficult to represent the three-dimensionsal chart, that one of the largest challenges of running the ρ-Bockstein spectral sequence is one of organization. We will explain some techniques for managing this.

5.2. **Sorting the E_1-page.** To analyze the ρ-Bockstein spectral sequence it is useful to sort the E_1-page by the Milnor-Witt degree $s - w$. The ρ-Bockstein differentials all have degree $(-1, 1, 0)$ with respect to the (s, f, w)-grading, and therefore have degree -1 with respect to the Milnor-Witt degree.

Table 1 shows the multiplicative generators for the ρ-Bockstein E_1-page through Milnor-Witt degree 5. The information in Table 1 was extracted from the $\mathrm{Ext}_{\mathbb{C}}$ chart in Figure 1 in the following manner. Lemma 2.2 says that elements in Milnor-Witt degree t satisfy $f \geq s - 2t$. Specifically, elements in Milnor-Witt degree at most 5 lie on or above the line $f = s - 10$ of slope 1.

This region is infinite, and in principle could contain generators in very high stems. However, in $\mathrm{Ext}_{\mathbb{C}}$ there is a line of slope 1/2 above which all elements are multiples of h_1 [5]. The line of slope 1 and the line of slope 1/2 bound a finite region which is easily searched exhaustively for generators of Milnor-Witt degree at most 5.

Note that the converse does not hold: some elements bounded by these lines may have Milnor-Witt degree greater than 5.

Our E_1-page is additively generated by all nonvanishing products of the elements from Table 1. Because the Bockstein differentials are ρ-linear, it suffices to understand how the differentials behave on products that do not involve ρ. Table 3 shows $\mathbb{F}_2[\rho]$-module generators for the E_1-page, sorted by Milnor-Witt degree.

5.3. **Bockstein differentials.**

Proposition 3.2 established some ρ-Bockstein differentials with a brute force approach via the cobar complex. We will next describe a different technique that computes all differentials in a large range.

All of our arguments will center on the ρ-local calculation of Theorem 4.1. This result says that if we invert ρ, then the ρ-Bockstein spectral sequence converges to a copy of $\mathrm{Ext}_{\mathrm{cl}} \otimes_{\mathbb{F}_2} \mathbb{F}_2[\rho, \rho^{-1}]$, with the motivic h_i corresponding to the classical h_{i-1}.

When identifying possible ρ-Bockstein differentials, there are two useful things to keep in mind:

- With respect to our $\mathrm{Ext}_{\mathbb{C}}$ chart, the differentials all go up one spot and left one spot;
- With respect to Table 3, the differentials all go to the left one column.

Combining these two facts (which involves switching back and forth between the chart and table), one can often severely narrow the possibilities for differentials.

Lemma 5.4. *The ρ-Bockstein d_1 differential is completely determined by:*

(1) $d_1(\tau) = \rho h_0$.
(2) The elements h_0, h_1, h_2, h_3, and c_0 are all permanent cycles.
(3) $d_1(Ph_1) = 0$.

Proof. The differential $d_1(\tau) = \rho h_0$ was established in Proposition 3.2.

The classes h_0 and h_1 cannot support differentials because there are no elements in negative Milnor-Witt degrees. The classes h_2 and h_3 must survive the ρ-local spectral sequence, so they cannot support differentials. Comparing chart and table, there are no possibilities for a differential on c_0.

Finally, if $d_1(Ph_1)$ is nonzero, then it is of the form ρx for a class x that does not contain ρ. This class x would appear at coordinates $(9, 6)$ in the $\mathrm{Ext}_{\mathbb{C}}$ chart. By inspection, there is no such x. □

Remark 5.5. We have shown that Ph_1 survives to the E_2-page, but we have not shown that it is a permanent cycle. The $\mathrm{Ext}_{\mathbb{C}}$ chart shows that $\rho^3 h_1^3 c_0$ is the only potential target for a differential on Ph_1. If Ph_1 is not a permanent cycle, then the only possibility is that $d_3(Ph_1)$ equals $\rho^3 h_1^3 c_0$. We will see below in Lemma 5.7 that this differential does occur.

Lemma 5.4 allows us to compute all d_1-differentials, using the product structure. Figure 2 displays the resulting E_2-page, sorted by Milnor-Witt degree.

Table 4 gives $\mathbb{F}_2[\rho]$-module generators for part of the E_2-page. Recall from Lemma 3.4 that ρ-torsion elements cannot be involved in any further differentials, so we have not included such elements in the table. We have also eliminated the elements that cannot be involved in any differentials because we know they are ρ-local by Theorem 4.1.

Note that τh_1 is indecomposable in the E_2-page, although τh_1^2 does decompose as $\tau h_1 \cdot h_1$. The multiplicative generators for the E_2-page are then

$$\boxed{h_0},\ \boxed{h_1},\ \tau h_1,\ \boxed{h_2},\ \tau^2,\ \tau h_2^2,\ \boxed{h_3},\ \boxed{c_0},\ \tau h_0^3 h_3,\ \tau c_0,\ Ph_1,$$

where boxes indicate classes that we already know are permanent cycles.

Lemma 5.6. *The ρ-Bockstein d_2 differential is completely determined by:*

(1) $d_2(\tau^2) = \rho^2 \cdot \tau h_1$.
(2) The elements τh_1, τh_2^2, and τc_0 are permanent cycles.
(3) $d_2(\tau h_0^3 h_3) = 0$.
(4) $d_2(Ph_1) = 0$.

Proof. The differential $d_2(\tau^2) = \rho^2 \tau h_1$ was established in Proposition 3.2.

Comparison of chart and table shows that a Bockstein differential on τh_1 could only hit h_0^2 or $\rho^2 h_1^2$. The first is impossible since the target of a d_2 differential must be divisible by ρ^2, and the second is ruled out by the fact that h_1^2 survives ρ-localization. So no differential can ever exist on τh_1.

Similarly, chart and table show that there are no possible differentials on τh_2^2, and no possible d_2 differential on either $\tau h_0^3 h_3$ or Ph_1.

It remains to consider τc_0. The only possibility for a differential is that $d_2(\tau c_0)$ might equal $\rho^2 h_1 c_0$. But if this happened we would also have $d_2(h_1^2 \tau c_0) = \rho^2 h_1^3 c_0$, which contradicts the fact that $\tau h_1^2 c_0$ is zero on the E_2-page, while $\rho^2 h_1^3 c_0$ is non-zero. □

Once again, Lemma 5.6 allows the complete computation of the E_3-page (in our given range), which is shown in Figure 2, sorted by Milnor-Witt degree. Table 5 gives $\mathbb{F}_2[\rho]$-module generators for part of the E_3-page. Recall from Lemma 3.4 that ρ-torsion elements cannot be involved in any further differentials, so we have not included such elements in the table. We have also eliminated the elements that cannot be involved in any differentials because we know they are ρ-local by Theorem 4.1.

The multiplicative generators for the E_3-page are

$$\boxed{h_0},\ \boxed{h_1},\ \boxed{\tau h_1},\ \boxed{h_2},\ \tau^2 h_0,\ \tau^2 h_2,\ \boxed{\tau h_2^2},\ \boxed{h_3},\ \boxed{c_0},\ \tau^4,\ \tau h_0^3 h_3,\ \boxed{\tau c_0},\ Ph_1,$$

where boxes indicate classes that we already know are permanent cycles.

Lemma 5.7. *The ρ-Bockstein d_3 differential is completely determined by:*

(1) $d_3(Ph_1) = \rho^3 h_1^3 c_0$.
(2) The elements $\tau^2 h_0$ and $\tau^2 h_2$ are permanent cycles.
(3) $d_3(\tau^4) = 0$.
(4) $d_3(\tau h_0^3 h_3) = 0$.

Proof. As we saw in Lemma 5.4, h_1 and c_0 are permanent cycles. Therefore, $h_1^3 c_0$ is a permanent cycle. We know from Theorem 4.1 that $h_1^3 c_0$ does not survive ρ-localization. Therefore, some differential hits $\rho^r h_1^3 c_0$. The only possibility is that $d_3(Ph_1)$ equals $\rho^3 h_1^3 c_0$.

Inspection of the E_3-page shows that there are no possible values for differentials on $\tau^2 h_0$. For $\tau^2 h_2$, there is a possibility that $d_4(\tau^2 h_2)$ equals $\rho^2 h_2^2$. However, this differential is ruled out by Theorem 4.1.

By inspection, there are no possible values for d_3 differentials on τ^4 or $\tau h_0^3 h_3$. □

The d_3 differential has a very mild effect on the E_3-page of our spectral sequence. In Table 5, the elements Ph_1 and $h_1^k Ph_1$ disappear from column four, the elements $h_1^k c_0$ disappear from column three for $k \geq 3$. Everything else remains the same, so we will not include a separate table for the E_4-page. The multiplicative generators are the same as for the E_3-page, except that Ph_1 is thrown out. Figure 2 depicts the E_4-page, sorted by Milnor-Witt degree.

Also, all of these generators are permanent cycles except possibly for τ^4 and $\tau h_0^3 h_3$. In particular, every element of the E_4-page in Milnor-Witt degrees strictly less than 4 is now known to be a permanent cycle. All the remaining differentials will go from Milnor-Witt degree 4 to Milnor-Witt degree 3.

Lemma 5.8. *The ρ-Bockstein d_4 differential is completely determined by:*

(1) $d_4(\tau^4) = \rho^4 \tau^2 h_2$.
(2) $d_4(\tau h_0^3 h_3) = \rho^4 h_1^2 c_0$.
(3) The other generators of the E_4-page are permanent cycles.

Proof. The differential $d_4(\tau^4) = \rho^4 \tau^2 h_2$ was established in Proposition 3.2.

We know that $h_1^2 c_0$ is a permanent cycle, but we also know from Theorem 4.1 that $h_1^2 c_0$ does not survive ρ-localization. Therefore, some differential hits $\rho^r h_1^2 c_0$ for some r. Looking at the chart, the only possibility is that $d_4(\tau h_0^3 h_3)$ equals $\rho^4 h_1^2 c_0$. □

The multiplicative generators of the E_5-page are the permanent cycles we have seen already, together with $\tau^4 h_0$ and $\tau^4 h_1$.

Lemma 5.9. *The ρ-Bockstein d_5 differential is zero.*

Proof. We only have to check for possible d_5 differentials on $\tau^4 h_0$ and $\tau^4 h_1$. Inspection of the $\mathrm{Ext}_{\mathbb{C}}$ chart shows that there are no classes in the relevant degrees. □

Figure 2 displays the E_6-page, sorted by Milnor-Witt degree.

Lemma 5.10. *The ρ-Bockstein d_6 differential is completely determined by:*

(1) $d_6(\tau^4 h_1) = \rho^6 \tau h_2^2$.
(2) The element $\tau^4 h_0$ is a permanent cycle.

Proof. Lemma 3.4 implies that $\tau^4 h_0$ is a permanent cycle because of the differential $d_1(\tau^5) = \rho \tau^4 h_0$.

By Theorem 4.1, we know that $\tau^4 h_1$ does not survive ρ-localization. Since $\rho^r \tau^4 h_1$ cannot be hit by a differential, it follows that $\tau^4 h_1$ supports a differential. The two possibilities are that $d_6(\tau^4 h_1)$ equals $\rho^6 \tau h_2^2$ or $d_8(\tau^4 h_1)$ equals $\rho^8 h_1 h_3$. We know from Theorem 4.1 that $h_1 h_3$ survives ρ-localization. Therefore, we must have $d_6(\tau^4 h_1) = \rho^6 \tau h_2^2$. □

The multiplicative generators for the E_7-page are $\tau^4 h_1^2$, together with other classes that we already know are permanent cycles. Figure 2 displays the E_7-page, sorted by Milnor-Witt degree.

Lemma 5.11. *The ρ-Bockstein d_7 differential is completely determined by:*

(1) $d_7(\tau^4 h_1^2) = \rho^7 c_0$.

Proof. By Theorem 4.1, we know that $\tau^4 h_1^2$ does not survive ρ-localization, and since $\rho^r \tau^4 h_1^2$ cannot be hit by a differential it follows that $\tau^4 h_1^2$ supports a differential. The two possibilities are that $d_7(\tau^4 h_1^2)$ equals $\rho^7 c_0$ or $d_8(\tau^4 h_1^2)$ equals $\rho^8 h_1^2 h_3$. We know from Theorem 4.1 that $h_1^2 h_3$ survives ρ-localization. Therefore, we must have $d_7(\tau^4 h_1^2) = \rho^7 c_0$. □

Finally, once we reach the E_8-page, we simply observe that all the multiplicative generators are classes that have already been checked to be permanent cycles.

5.12. **The ρ-Bockstein E_∞-page.** Table 6 describes the ρ-Bockstein E_∞-page in the range of interest. The table gives a list of $\mathbb{M}_2$-module generators for the E_∞-page. We write $x\,(\rho^k)$ if x is killed by ρ^k, and we write $x(\mathrm{loc})$ for classes that are non-zero after ρ-localization.

The reader is invited to construct a single Adams chart that captures all of this information. We have found that combining all of the Milnor-Witt degrees into one picture makes it too difficult to get a feel for what is going on. For example, at coordinates $(3, 3)$, one has six elements h_1^3, τh_1^3, $\tau^2 h_1^3$, $\tau^3 h_1^3$, $\rho^5 c_0$, and $\rho^6 h_1^2 h_3$. Each of these elements is related by h_0, h_1, and ρ extensions to other elements.

6. From the ρ-Bockstein E_∞-page to $\mathrm{Ext}_\mathbb{R}$

Having obtained the E_∞-page of the ρ-Bockstein spectral sequence, we will now compute all hidden extensions in the range under consideration. The key arguments rely on May's Convergence Theorem [11] in a slightly unusual way. We will use this theorem to argue that certain Massey products $\langle a, b, c\rangle$ cannot be well-defined. We will deduce that either ab or bc must be non-zero via a hidden extension.

Remark 6.1. As is typical in this kind of analysis, there are issues underlying the naming of classes. An element x of the Bockstein E_∞-page represents a coset of elements of $\mathrm{Ext}_\mathbb{R}$, and it is convenient if we can slightly ambiguously use the same symbol x for one particular element from this coset. This selection has to happen on a case-by-case basis, but once done it allows us to use the same symbols for elements of the Bockstein E_∞-page and for elements of $\mathrm{Ext}_\mathbb{R}$ that they represent.

For example, the element h_0 on the E_∞-page represents two elements of $\mathrm{Ext}_\mathbb{R}$, because of the presence of ρh_1 in higher Bockstein filtration. One of these elements is annihilated by ρ and the other is not. We write h_0 for the element of $\mathrm{Ext}_\mathbb{R}$ that is annihilated by ρ.

Table 8 summarizes these ambiguities and gives definitions in terms of ρ-torsion.

Once again, careful bookkeeping is critical at this stage. We begin by choosing preferred $\mathbb{F}_2[\rho]$-module generators for $\mathrm{Ext}_\mathbb{R}$ up to Milnor-Witt degree 4. First, we choose an ordering of the multiplicative generators of $\mathrm{Ext}_\mathbb{R}$:

$$\rho < h_0 < h_1 < \tau h_1 < h_2 < \tau^2 h_0 < \tau^2 h_2 < \tau h_2^2 < h_3 < c_0 < \tau^4 h_0 < \tau c_0.$$

The ordering here is essentially arbitrary, although it is convenient to have elements of low Milnor-Witt degree appear first.

Next, we choose $\mathbb{F}_2[\rho]$-module generators for $\mathrm{Ext}_\mathbb{R}$ that come first in the lexicographic ordering on monomials in these generators. For example, we could choose either $h_0^2 h_2$ or $\tau h_1 \cdot h_1^2$ to be an $\mathbb{F}_2[\rho]$-module generator; we select $h_0^2 h_2$ because $h_0 < h_1$. We do this for each element listed in Table 6.

The results of these choices are displayed in Table 7. This table lists $\mathbb{F}_2[\rho]$-module generators of $\mathrm{Ext}_\mathbb{R}$. We write $x\,(\rho^k)$ if x is killed by ρ^k, and we write $x(\mathrm{loc})$ for classes that are non-zero after ρ-localization.

Our goal is to produce a list of relations for $\mathrm{Ext}_\mathbb{R}$ that allows every monomial to be reduced to a linear combination of monomials listed in Table 7. We will begin by considering all pairwise products of generators.

Lemma 6.2. *Through Milnor-Witt degree 4, Table 9 lists the products of all pairs of multiplicative generators of* $\mathrm{Ext}_\mathbb{R}$.

In Table 9, the symbol $-$ indicates that the product has no simpler form, i.e., is a monomial listed in Table 7.

Proof. Some products are zero because there is no other possbility; for example $h_1 h_2$ is zero because there are no non-zero elements in the appropriate degree.

Some products are zero because we already know that they are annihilated by some power of ρ, while the only non-zero elements in the appropriate degree are all ρ-local. For example, for degree reasons, it is possible that $h_0 h_1$ equals ρh_1^2. However, we already know that ρh_0 is zero, while h_1^2 is ρ-local. Therefore, $h_0 h_1$ must be zero. Similar arguments explain all of the pairwise products that are zero in Table 9.

Some of the non-zero pairwise products are not hidden in the ρ-Bockstein spectral sequence. For example, consider the product $\tau^2 h_0 \cdot h_2$. We have that $\tau^2 h_0 \cdot h_2 + h_0 \cdot \tau^2 h_2$ is zero on the ρ-Bockstein E_∞-page, but $\tau^2 h_0 \cdot h_2 + h_0 \cdot \tau^2 h_2$ might equal something of higher ρ-filtration in $\mathrm{Ext}_\mathbb{R}$. The possible values for this expression in $\mathrm{Ext}_\mathbb{R}$ are the linear combinations of $\rho^3 \cdot \tau h_2^2$ and $\rho^5 h_1 h_3$. Both of these elements are non-zero after multiplication by ρ, while $\tau^2 h_0 \cdot h_2 + h_0 \cdot \tau^2 h_2$ is annihilated by ρ in $\mathrm{Ext}_\mathbb{R}$. Therefore, we must have that $\tau^2 h_0 \cdot h_2 + h_0 \cdot \tau^2 h_2 = 0$ in $\mathrm{Ext}_\mathbb{R}$.

The same argument applies to the other non-hidden extensions in Table 9, except that they are somewhat easier because there are no possible hidden values.

The remaining non-zero pairwise products are all hidden in the ρ-Bockstein spectral sequence. For these, we need a more sophisticated argument involving Massey products and May's Convergence Theorem [11]. This theorem says that under certain technical conditions involving the vanishing of "crossing" differentials, one can compute Massey products in $\mathrm{Ext}_\mathbb{R}$ using the ρ-Bockstein differentials.

We will demonstrate how this works for the product $h_0 \cdot \tau h_1$. Consider the Massey product $\langle \rho, h_0, \tau h_1 \rangle$ in $\mathrm{Ext}_\mathbb{R}$. If this Massey product were well-defined, then May's Convergence Theorem and the ρ-Bockstein differential $d_1(\tau) = \rho h_0$ would imply that the Massey product contains an element that is detected by $\tau^2 h_1$ in the ρ-Bockstein E_∞-page. (Beware that one needs to check that there are no crossing differentials.) The element $\tau^2 h_1$ does not survive to the E_∞-page. Therefore, the Massey product is not well-defined, so $h_0 \cdot \tau h_1$ must be non-zero. The only possible value for the product is $\rho h_1 \cdot \tau h_1$.

The same style of argument works for all of the hidden extensions listed in Table 9, with one additional complication in some cases. Consider the product $h_1 \cdot \tau^2 h_0$.

Analysis of the Massey product $\langle \rho, \tau^2 h_0, h_1 \rangle$ implies that the product must be non-zero, since $\tau^3 h_1$ does not survive to the ρ-Bockstein E_∞-page. However, there is more than one possible value for $h_1 \cdot \tau^2 h_0$; it could be any linear combination of $\rho(\tau h_1)^2$ and $\rho^5 h_2^2$. We know that $\rho \cdot \tau^2 h_0$ is zero, while h_2^2 is ρ-local. Therefore, we deduce that $h_1 \cdot \tau^2 h_0$ equals $\rho(\tau h_1)^2$. This type of ρ-local analysis allows us to nail down the precise value of each hidden extension in every case where there is more than one possible non-zero value. □

Proposition 6.3. *Table 10 gives some relations in* $\mathrm{Ext}_{\mathbb{R}}$ *that are hidden in the* ρ*-Bockstein spectral sequence. These relations, together with the products given in Table 9, are a complete set of multiplicative relations for* $\mathrm{Ext}_{\mathbb{R}}$ *up to Milnor-Witt degree* 4*.*

Proof. These relations follow from the same types of arguments that are given in the proof of Lemma 6.2. The most interesting is the relation $h_0^2 \cdot \tau^2 h_2 + (\tau h_1)^3 = \rho^5 c_0$, which follows from an analysis of the matric Massey product

$$\left\langle \rho^2, \begin{bmatrix} h_0 & \tau h_1 \end{bmatrix}, \begin{bmatrix} h_0 \cdot \tau^2 h_2 \\ (\tau h_1)^2 \end{bmatrix} \right\rangle.$$

If this matric Massey product were defined, then May's Convergence Theorem and the differential $d_2(\tau^2) = \rho^2 \tau h_1$ would imply that it is detected by $\tau^4 h_1^2$ in the ρ-Bockstein E_∞-page. But $\tau^4 h_1^2$ does not survive to the ρ-Bockstein E_∞-page.

For every monomial x in Table 7 and every multiplicative generator y of $\mathrm{Ext}_{\mathbb{R}}$, one can check by brute force that the relations in Tables 9 and 10 allow one to identify xy in terms of the monomials in Table 7. □

Figure 3 displays $\mathrm{Ext}_{\mathbb{R}}$, sorted by Milnor-Witt degree. The picture is similar to the E_∞-page shown in Figure 2, except that the hidden extensions by h_0 and by h_1 are indicated with dashed lines.

7. The Adams spectral sequence

At this point we have computed the tri-graded ring

$$\mathrm{Ext}_{\mathbb{R}} = \mathrm{Ext}_{\mathcal{A}}^{*,*,*}(\mathbb{M}_2, \mathbb{M}_2)$$

up through Milnor-Witt degree four. We will now consider the motivic Adams spectral sequence based on mod 2 motivic cohomology, which takes the form

$$\mathrm{Ext}_{\mathcal{A}}^{s,f,w}(\mathbb{M}_2, \mathbb{M}_2) \Rightarrow \hat{\pi}_{s,w}.$$

Recall that we are writing $\hat{\pi}_{*,*}$ for the motivic stable homotopy groups of the completion of the motivic sphere spectrum with respect to the motivic Eilenberg-Mac Lane spectrum $H\mathbb{F}_2$. The Adams d_r differential takes elements of tridegree (s, f, w) to elements of tridegree $(s-1, f+r, w)$. In particular, the Adams d_r differential decreases the Milnor-Witt degree by 1. So it pays off to once again fracture the E_2-page into the different Milnor-Witt degrees.

It turns out that there are no Adams differentials in the range under consideration, as shown in the following result.

Proposition 7.1. *Up through Milnor-Witt degree four, there are no differentials in the motivic Adams spectral sequence.*

Proof. The proof uses Table 7 and the $\mathrm{Ext}_{\mathbb{R}}$ charts in Figure 3 to keep track of elements.

The elements ρ, h_0, and h_1 are permanent cycles, as there are no classes in Milnor-Witt degree -1. For $\tau^2 h_0$, we observe that there are no classes of Milnor-Witt degree 1 in the range of the possible differentials on $\tau^2 h_0$. Similarly, there are no possible values in Milnor-Witt degree 2 for differentials on τh_2^2, h_3, and c_0.

For degree reasons, the only possible values for $d_r(\tau h_1)$ are h_0^{r+1} and $\rho^{r+1} h_1^{r+1}$. However, $h_0^2 \cdot h_0^{r+1}$ is non-zero on the Adams E_r-page, while $h_0^2 \cdot \tau h_1$ is zero. Also, $\rho^2 \cdot \rho^{r+1} h_1^{r+1}$ is non-zero on the Adams E_r-page, while $\rho^2 \cdot \tau h_1$ is zero. This implies that there are no differentials on τh_1.

The only possible value for $d_r(h_2)$ is $\rho^{r-1} h_1^{r+1}$. However, $h_1 \cdot \rho^{r-1} h_1^{r+1}$ is non-zero on the Adams E_r-page, while $h_1 \cdot h_2$ is zero. This implies that there are no differentials on h_2.

The only possibility for a nonzero differential on $\tau^4 h_0$ is that $d_2(\tau^4 h_0)$ might equal to $\rho^{10} h_1^2 h_3$. However, $\rho \cdot \tau^4 h_0$ is zero on the Adams E_2-page, while $\rho \cdot \rho^{10} h_1^2 h_3$ is not. This implies that there are no differentials on $\tau^4 h_0$.

It remains to show that $\tau^2 h_2$ and τc_0 are permanent cycles. We handle these more complicated arguments below in Lemmas 7.3 and 7.6. □

Lemma 7.2. *The Massey product $\langle \rho^2, \tau h_1, h_2 \rangle$ contains $\tau^2 h_2$, with indeterminacy generated by $\rho^4 h_3$.*

Proof. Apply May's Convergence Theorem [11], using the ρ-Bockstein differential $d_2(\tau^2) = \rho^2 \cdot \tau h_1$. This shows that $\tau^2 h_2$ or $\tau^2 h_2 + \rho^4 h_3$ is contained in the bracket. By inspection, the indeterminacy is generated by $\rho^4 h_3$. □

Lemma 7.3. *The element $\tau^2 h_2$ is a permanent cycle.*

Proof. As shown in Table 11, let $\tau\eta$ and ν be elements of $\hat{\pi}_{1,0}$ and $\hat{\pi}_{3,2}$ respectively that are detected by τh_1 and h_2. The product $\rho^2 \cdot \tau\eta$ is zero because there is no other possibility. For degree reasons, the product $\tau\eta \cdot \nu$ could possibly equal $\rho^2 \nu^2$. However, $\rho^2 \cdot \tau\eta \cdot \nu$ is zero, while $\rho^2 \cdot \rho^2 \nu^2$ is not. Therefore, $\tau\eta \cdot \nu$ is also zero.

We have just shown that the Toda bracket $\langle \rho^2, \tau\eta, \nu \rangle$ is well-defined. Moss's Convergence Theorem [16] then implies that the Massey product $\langle \rho^2, \tau h_1, h_2 \rangle$ contains a permanent cycle. We computed this Massey product in Lemma 7.2, so we know that $\tau^2 h_2$ or $\tau^2 h_2 + \rho^4 h_3$ is a permanent cycle. We already know that $\rho^4 h_3$ is a permanent cycle, so $\tau^2 h_2$ is also a permanent cycle. □

For completeness, we will give an alternative proof that $\tau^2 h_2$ is a permanent cycle that has a more geometric flavor. There is a functor from classical homotopy theory to motivic homotopy theory over $\mathbb{R}$ (or over any field) that takes the sphere S^p to $S^{p,0}$. Let ν_{top} be the unstable map $S^{7,0} \to S^{4,0}$ that is the image under this functor of the classical Hopf map $S^7 \to S^4$.

Lemma 7.4. *The cohomology of the cofiber of ν_{top} is a free $\mathbb{M}_2$-module on two generators x and y of degrees $(4,0)$ and $(8,0)$, satisfying $\mathrm{Sq}^4(x) = \tau^2 y$ and $\mathrm{Sq}^8(x) = \rho^4 y$.*

Proof. Consider the cofiber sequence

$$S^{7,0} \to S^{4,0} \to C\nu_{\mathrm{top}} \to S^{8,0},$$

where $C\nu_{\mathrm{top}}$ is the cofiber of ν_{top}. Apply motivic cohomology to obtain a long exact sequence. It follows that the cohomology of $C\nu_{\mathrm{top}}$ is a free $\mathbb{M}_2$-module on two generators x and y of degrees $(4,0)$ and $(8,0)$.

For degree reasons, the only possible non-zero cohomology operations are that $\mathrm{Sq}^4(x)$ and $\mathrm{Sq}^8(x)$ might equal $\tau^2 y$ and $\rho^4 y$ respectively. The formula $\mathrm{Sq}^4(x) = \tau^2 y$ follows by comparison to the classical case.

The formula for $\mathrm{Sq}^8(x)$ is more difficult. Consider $S^{4,4} \wedge C\nu_{\mathrm{top}}$, which has cells in dimensions $(8,4)$ and $(12,8)$. The cohomology generator in degree $(8,4)$ is the external product $z \wedge x$, where z is the cohomology generator of $S^{4,4}$ in degree $(4,4)$. The cohomology generator in degree $(12,4)$ is $z \wedge y$.

Now we can compute Sq^8 in terms of the cup product $(z \wedge x)^2$. According to [20, Lemma 6.8], the cup product z^2 equals $\rho^4 z$ in the cohomology of $S^{4,4}$. Also, the cup product x^2 equals y in the cohomology of $C\nu_{\mathrm{top}}$ by comparison to the classical case. By the Künneth formula, it follows that $(z \wedge x)^2 = \rho^4(z \wedge y)$ and that $\mathrm{Sq}^8(x) = \rho^4 y$. □

Another proof of Lemma 7.3. Lemma 7.4 shows that the stabilization of ν_{top} in $\hat{\pi}_{3,0}$ is detected by $\tau^2 h_2 + \rho^4 h_3$ in the motivic Adams spectral sequence. There are elements ρ and σ in $\hat{\pi}_{-1,-1}$ and $\hat{\pi}_{7,4}$ detected by ρ and h_3 in the motivic Adams spectral sequence. Therefore, $\tau^2 h_2$ is a permanent cycle that detects $\nu_{\mathrm{top}} + \rho^4\sigma$. □

Lemma 7.5. *The Massey product $\langle \tau h_1, h_2, h_0 h_2\rangle$ contains τc_0, with indeterminacy generated by $\rho \cdot \tau h_1 \cdot h_1 h_3$.*

Proof. Recall that there is a classical Massey product $\langle h_1, h_2, h_0 h_2\rangle = c_0$. This implies that the motivic Massey product $\langle \tau h_1, h_2, h_0 h_2\rangle$ contains τc_0.

By inspection, the indeterminacy is generated by $\rho \cdot \tau h_1 \cdot h_1 h_3$. □

Lemma 7.6. *The element τc_0 is a permanent cycle.*

Proof. As shown in Table 11, let $\tau\eta$ and ω be elements of $\hat{\pi}_{1,0}$ and $\hat{\pi}_{0,0}$ detected by τh_1 and h_0. As in the proof of Lemma 7.3, the product $\tau\eta \cdot \nu$ is zero. Also, the product $\omega\nu^2$ is zero because there are no other possibilities.

We have just shown that the Toda bracket $\langle \tau\eta, \nu, \omega\nu\rangle$ is well-defined. Moss's Convergence Theorem [16] then implies that the Massey product $\langle \tau h_1, h_2, h_0 h_2\rangle$ contains a permanent cycle. We computed this Massey product in Lemma 7.5, so we know that τc_0 or $\tau c_0 + \rho \cdot \tau h_1 \cdot h_1 h_3$ is a permanent cycle. We already know that $\rho \cdot \tau h_1 \cdot h_1 h_3$ is a permanent cycle, so τc_0 is also a permanent cycle. □

8. Milnor-Witt modules and $\hat{\pi}_{*,*}$

In this section, we will describe how to pass from the Adams E_∞-page to $\hat{\pi}_{*,*}$. We recall the following well-known elements [4] [13].

1. ϵ in $\hat{\pi}_{0,0}$ is represented by the twist map on $S^{1,1} \wedge S^{1,1}$.
2. ρ in $\hat{\pi}_{-1,-1}$ is represented by the inclusion $\{\pm 1\} \to (\mathbb{A}^1 - 0)$.
3. η in $\hat{\pi}_{1,1}$ is represented by the Hopf construction on the multiplication $(\mathbb{A}^1 - 0) \times (\mathbb{A}^1 - 0) \to (\mathbb{A}^1 - 0)$.
4. ν in $\hat{\pi}_{3,2}$ is represented by the Hopf construction on a version of quaternionic multiplication.
5. σ in $\hat{\pi}_{7,4}$ is represented by the Hopf construction on a version of octonionic multiplication.

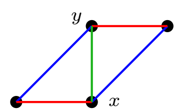
The element $1 - \epsilon$ is detected in the Adams spectral sequence by h_0. Thus it, rather than 2, deserves to be considered the zeroth motivic Hopf map. Because it plays a critical role, it is convenient to give this element a name.

Definition 8.1. The element ω in $\hat{\pi}_{0,0}$ equals $1 - \epsilon$.

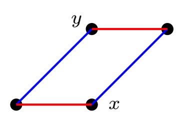
The motivic Adams E_∞-page is the associated graded module of the motivic homotopy groups $\hat{\pi}_{*,*}$ with respect to the adic filtration for the ideal generated by ω and η. Note that this ideal also equals $(2, \eta)$ because of the relation $\rho\eta = -1-\epsilon = \omega - 2$.

The elements ρ, h_0, and h_1 detect the homotopy elements ρ, ω, and η in the 0th Milnor-Witt stem Π_0. The relation $2 = \omega - \rho\eta$ implies that $h_0 + \rho h_1$, rather than h_0, detects 2. This means that we must be careful when computing the additive structure of Milnor-Witt stems.

In the Adams chart, a parallelogram such as



indicates that 2 times the homotopy elements detected by x are zero (or detected in higher Adams filtration by a hidden extension) because $(h_0 + \rho h_1)x = 0$. On the other hand, a parallelogram such as



indicates that 2 times the homotopy elements detected by x are detected by y because $(h_0 + \rho h_1)x = y$.

We will choose specific homotopy elements to serve as our Π_0-module generators. Because of the associated graded nature of the Adams E_∞-page, there is some choice in these generators. For the most part, the Π_0-module structures of the Milnor-Witt modules in our range are insensitive to these choices, so this is not of immediate concern. However, we would like to be as precise as we can to facilitate further study.

These observations allow us to pass from the Adams spectral sequence to the diagrams of Π_0, Π_1, Π_2, and Π_3 given in Figure 4.

8.2. **The zeroth Milnor-Witt module.** For Π_0, the Adams spectral sequence consists of an infinite sequence of dots extending upwards in each stem except for the 0-stem. These dots are all connected by 2 extensions, so they assemble into copies of $\mathbb{Z}_2$. The 0-stem is somewhat more complicated. Here there are two sequences of dots extending upwards: elements of the form $\rho^k h_1^k$ and elements of the form $(h_0 + \rho h_1)^k = h_0^k + \rho^k h_1^k$. The former elements assemble into a copy of $\mathbb{Z}_2$ generated by $\rho\eta$, while the latter elements assemble into a copy of $\mathbb{Z}_2$ generated by 1.

8.3. **The first Milnor-Witt module.** For Π_1, there are three elements in the 3-stem of the Adams spectral sequence. These elements assemble into a copy of

$\mathbb{Z}/8$ generated by ν; note that $h_0^k h_2 = (h_0 + \rho h_1)^k h_2$ because $h_1 h_2 = 0$. The two elements τh_1 and $\rho\tau h_1^2$ in the 1-stem do not assemble into a copy of $\mathbb{Z}/4$ because $(h_0 + \rho h_1)\tau h_1$ is zero.

We will now discuss precise definitions of the Π_0-module generators of Π_1.

Recall that there is a functor from classical homotopy theory to motivic homotopy theory over $\mathbb{R}$ that takes a sphere S^p to $S^{p,0}$. Let η_{top} in $\hat{\pi}_{1,0}$ be the image of the classical Hopf map η. By an argument analogous to the proof of Lemma 7.4, η_{top} is detected by $\tau h_1 + \rho^2 h_2$. Therefore $\eta_{\mathrm{top}} + \rho^2\nu$ is detected by τh_1.

Definition 8.4. Let $\tau\eta$ be the element $\eta_{\mathrm{top}} + \rho^2\nu$ of $\hat{\pi}_{1,0}$.

Another possible approach to defining $\tau\eta$ is to use a Toda bracket to specify a single element. However, the obvious Toda brackets detecting τh_1 all have indeterminacy, so they are unsuitable for this purpose.

In terms of algebraic formulas we could write

$$\Pi_1 = \Pi_0\langle \tau\eta, \nu\rangle/(2\cdot\tau\eta, 8\nu, \eta\nu, \rho^2\cdot\tau\eta, \eta^2\cdot\tau\eta - 4\nu),$$

but we find Figure 4 to be more informative.

8.5. **The second Milnor-Witt module.** The calculation of Π_2 involves the same kinds of considerations that we already described for Π_0 and for Π_1. The names of the generators $(\tau\eta)^2$ and ν^2 reflect the multiplicative structure of the Milnor-Witt stems.

It remains to specify a choice of generator detected by $\tau^2 h_0$. Recall that there is a realization functor from motivic homotopy theory over $\mathbb{R}$ to classical homotopy theory. (This functor factors through $\mathbb{Z}/2$-equivariant homotopy theory, but we won't use the equivariance for now.)

Definition 8.6. Let $\tau^2\omega$ be the element of $\hat{\pi}_{0,-2}$ detected by $\tau^2 h_0$ that realizes to 2 in classical π_0.

In terms of algebraic formulas we could write

$$\Pi_2 = \Pi_0\langle\nu^2\rangle/(2\nu^2, \eta\nu^2) \oplus \Pi_0\langle\tau^2\omega, (\tau\eta)^2\rangle/(\rho\cdot\tau^2\omega, 2(\tau\eta)^2, \eta^2(\tau\eta)^2, \rho(\tau\eta)^2 - \eta\cdot\tau^2\omega).$$

The unreadability of this formula illustrates why the graphical calculus of Figure 4 is so helpful.

8.7. **The third Milnor-Witt module.** The structure of Π_3 is significantly more complicated. We will begin by discussing choices of generators.

Definition 8.8. Let $\tau^2\nu$ be the element $\nu_{\mathrm{top}} + \rho^4\sigma$ in $\hat{\pi}_{3,0}$.

A precise definition of the generator $\tau\nu^2$ detected by τh_2^2 has so far eluded us. For the purposes of this article, it suffices to choose an arbitrary homotopy element detected by τh_2^2. The distinction between the choices is not relevant in the range under consideration here, but it may be important for an analysis of higher Milnor-Witt stems.

Lemma 8.9 gives a definition of the generator detected by c_0, assuming that $\tau\nu^2$ has already been chosen.

Lemma 8.9. *There is a unique element $\overline{\epsilon}$ in $\hat{\pi}_{8,5}$ detected by c_0 such that $\rho\overline{\epsilon} - \eta\cdot\tau\nu^2$ equals zero.*

Proof. Let $\overline{\epsilon}'$ be any element detected by c_0. The relation $\rho c_0 + h_1 \cdot \tau h_2^2 = 0$ implies that $\rho\overline{\epsilon}' - \eta \cdot \tau\nu^2$ is detected in higher Adams filtration. Note that ω kills $\rho\overline{\epsilon}' - \eta \cdot \tau\nu^2$ because it kills both ρ and η. Therefore, $\rho\overline{\epsilon}' - \eta \cdot \tau\nu^2$ cannot be detected by $h_0^3 h_3$. If $\rho\overline{\epsilon}' - \eta \cdot \tau\nu^2$ is detected by $\rho^2 h_1 c_0$, then we can add an element detected by $\rho h_1 c_0$ to $\overline{\epsilon}'$ to obtain our desired element $\overline{\epsilon}$. Similarly, if $\rho\overline{\epsilon}' - \eta \cdot \tau\nu^2$ is detected by $\rho^3 h_1^2 c_0$, then we can add an element detected by $\rho^2 h_1^2 c_0$ to $\overline{\epsilon}'$. □

Remark 8.10. The classical analogue of $\overline{\epsilon}$ is traditionally called ϵ; we have changed the notation to avoid the unfortunate coincidence with the motivic element ϵ in $\hat{\pi}_{0,0}$.

Having determined generators for Π_3, we now proceed to analyze its Π_0-module structure. For the most part, this analysis follows the same arguments familiar from the earlier Milnor-Witt stems. The 3-stem and the 7-stem present the greatest challenges, so we discuss them in more detail.

In the 3-stem, $\tau^2\nu$ generates a copy of $\mathbb{Z}/8$; these eight elements are detected by $\tau^2 h_2$, $h_0 \cdot \tau^2 h_2 + \rho^3 \cdot \tau h_2^2$, and $h_0^2 \cdot \tau^2 h_2 + \rho^5 c_0$. The element $\rho^3 \cdot \tau\nu^2$ also generates a copy of $\mathbb{Z}/8$; these eight elements are detected by $\rho^3 \cdot \tau h_2^2$, $\rho^5 c_0$, and $\rho^6 h_1 c_0$. Finally, $\rho^4\sigma$ generates a copy of $\mathbb{Z}/8$; these eight elements are detected by $\rho^4 h_3$, $\rho^5 h_1 h_3$, and $\rho^6 h_1^2 h_3$.

In the 7-stem, σ generates a copy of $\mathbb{Z}/32$; these 32 elements are detected by h_3, $h_0 h_3 + \rho h_1 h_3$, $h_0^2 h_3 + \rho^2 h_1^2 h_3$, $h_0^3 h_3$, and $\rho^3 h_1^2 c_0$. The element $\rho\eta\sigma$ generates a copy of $\mathbb{Z}/4$; these four elements are detected by $\rho h_1 h_3$ and $\rho^2 h_1^2 h_3$. The element $\rho\overline{\epsilon} - 4\sigma$ also generates a copy of $\mathbb{Z}/4$; these four elements are detected by $h_0^2 h_3 + \rho^2 h_1^2 h_3 + \rho c_0$ and $h_0^3 h_3 + \rho^2 h_1 c_0$.

The η extension from $\eta^2\sigma$ to $\eta^2\overline{\epsilon}$ is hidden in the Adams spectral sequence. This is the same as the analogous hidden extension in the classical situation [18]. This is the only hidden extension by ρ, ω, or η in the range under consideration.

9. Tables

Table 1: Multiplicative generators for the ρ-Bockstein E_1-page

$s-w$	element	(s,f,w)
0	ρ	$(-1,0,-1)$
0	h_0	$(0,1,0)$
0	h_1	$(1,1,1)$
1	τ	$(0,0,-1)$
1	h_2	$(3,1,2)$
3	h_3	$(7,1,4)$
3	c_0	$(8,3,5)$
4	Ph_1	$(9,5,5)$
5	Ph_2	$(11,5,6)$

Table 2: Bockstein differentials

(s,f,w)	x	d_r	$d_r(x)$	proof
$(0,0,-1)$	τ	d_1	ρh_0	Lemma 5.4
$(0,0,-2)$	τ^2	d_2	$\rho^2\tau h_1$	Lemma 5.6
$(0,0,-4)$	τ^4	d_4	$\rho^4\tau^2 h_2$	Lemma 5.8
$(1,1,-3)$	$\tau^4 h_1$	d_6	$\rho^6\tau h_2^2$	Lemma 5.10
$(2,2,-2)$	$\tau^4 h_1^2$	d_7	$\rho^7 c_0$	Lemma 5.10
$(7,4,3)$	$\tau h_0^3 h_3$	d_4	$\rho^4 h_1^2 c_0$	Lemma 5.8
$(9,5,5)$	Ph_1	d_3	$\rho^3 h_1^3 c_0$	Lemma 5.7

Table 3: $\mathbb{F}_2[\rho]$-module generators for the ρ-Bockstein E_1-page

0	1	2	3		4	
h_0^k	τ	τ^2	τ^3	h_3	τ^4	$\tau h_0 h_3$
h_1^k	τh_0^k	$\tau^2 h_0^k$	$\tau^3 h_0^k$	$h_0 h_3$	$\tau^4 h_0^k$	$\tau h_0^2 h_3$
	τh_1	$\tau^2 h_1$	$\tau^3 h_1$	$h_0^2 h_3$	$\tau^4 h_1$	$\tau h_0^3 h_3$
	τh_1^2	$\tau^2 h_1^2$	$\tau^3 h_1^2$	$h_0^3 h_3$	$\tau^4 h_1^2$	$\tau h_1 h_3$
	τh_1^3	$\tau^2 h_1^3$	$\tau^3 h_1^3$	$h_1 h_3$	$\tau^4 h_1^3$	$\tau h_1^2 h_3$
	h_2	τh_2	$\tau^2 h_2$	$h_1^2 h_3$	$\tau^3 h_2$	τc_0
	$h_0 h_2$	$\tau h_0 h_2$	$\tau^2 h_0 h_2$	c_0	$\tau^3 h_0 h_2$	$\tau h_1 c_0$
		h_2^2	τh_2^2	$h_1^k c_0$	$\tau^2 h_2^2$	Ph_1
					τh_3	$h_1^k Ph_1$

Table 4: Some $\mathbb{F}_2[\rho]$-module generators for the ρ-Bockstein E_2-page

0	1	2	3	4	
	τh_1	τ^2	$\tau^3 h_1$	τ^4	$\tau h_1 h_3$
	τh_1^2	$\tau^2 h_1$	$\tau^3 h_1^2$	$\tau^4 h_1$	$\tau h_1^2 h_3$
		$\tau^2 h_1^2$	$\tau^2 h_2$	$\tau^4 h_1^2$	τc_0
		$\tau^2 h_1^3$	τh_2^2	$\tau^4 h_1^3$	$\tau h_1 c_0$
			c_0	$\tau^2 h_2^2$	$P h_1$
			$h_1^k c_0$	$\tau h_0^3 h_3$	$h_1^k P h_1$

Table 5: Some $\mathbb{F}_2[\rho]$-module generators for the ρ-Bockstein E_3-page

0	1	2	3	4	
			$\tau^2 h_2$	τ^4	$\tau h_0^3 h_3$
			τh_2^2	$\tau^4 h_1$	τc_0
			c_0	$\tau^4 h_1^2$	$P h_1$
			$h_1^k c_0$	$\tau^4 h_1^3$	$h_1^k P h_1$
				$\tau^2 h_2^2$	

Table 6: $\mathbb{F}_2[\rho]$-module generators for the ρ-Bockstein E_∞-page

0	1	2	3		4
$h_0^k\,(\rho)$	$\tau h_1\,(\rho^2)$	$\tau^2 h_0^k\,(\rho)$	$\tau^3 h_1^3\,(\rho)$	$h_0^3 h_3\,(\rho)$	$\tau^4 h_0^k\,(\rho)$
$h_1^k(\mathrm{loc})$	$\tau h_1^2\,(\rho^2)$	$\tau^2 h_1^2\,(\rho^2)$	$\tau^2 h_2\,(\rho^4)$	$h_1 h_3(\mathrm{loc})$	$\tau^2 h_2^2\,(\rho^4)$
	$\tau h_1^3\,(\rho)$	$\tau^2 h_1^3\,(\rho^2)$	$\tau^2 h_0 h_2\,(\rho)$	$h_1^2 h_3(\mathrm{loc})$	$\tau h_1 h_3\,(\rho^2)$
	$h_2(\mathrm{loc})$	$h_2^2(\mathrm{loc})$	$\tau h_2^2\,(\rho^6)$	$c_0\,(\rho^7)$	$\tau h_1^2 h_3\,(\rho^2)$
	$h_0 h_2\,(\rho)$		$h_3(\mathrm{loc})$	$h_1 c_0\,(\rho^7)$	$\tau c_0\,(\rho^3)$
			$h_0 h_3\,(\rho)$	$h_1^2 c_0\,(\rho^4)$	$\tau h_1 c_0\,(\rho^2)$
			$h_0^2 h_3\,(\rho)$	$h_1^{k+3} c_0\,(\rho^3)$	

Table 7: $\mathbb{F}_2[\rho]$-module generators for $\mathrm{Ext}_{\mathbb{R}}$

0	1	2	3		4
$h_0^k\,(\rho)$	$\tau h_1\,(\rho^2)$	$\tau^2 h_0\,(\rho)$	$\tau^2 h_2\,(\rho^4)$	$h_0^3 h_3\,(\rho)$	$\tau^4 h_0\,(\rho)$
$h_1^k(\mathrm{loc})$	$\tau h_1 \cdot h_1\,(\rho^2)$	$\tau^2 h_0 \cdot h_0^k\,(\rho)$	$h_0 \cdot \tau^2 h_2\,(\rho)$	$h_1 h_3(\mathrm{loc})$	$\tau^4 h_0 \cdot h_0^k\,(\rho)$
	$h_2(\mathrm{loc})$	$(\tau h_1)^2\,(\rho^2)$	$h_0^2 \cdot \tau^2 h_2\,(\rho)$	$h_1^2 h_3(\mathrm{loc})$	$\tau^2 h_2 \cdot h_2\,(\rho^4)$
	$h_0 h_2\,(\rho)$	$(\tau h_1^2) h_1\,(\rho^2)$	$\tau h_2^2\,(\rho^6)$	$c_0\,(\rho^7)$	$\tau h_1 \cdot h_3\,(\rho^2)$
	$h_0^2 h_2\,(\rho)$	$h_2^2(\mathrm{loc})$	$h_3(\mathrm{loc})$	$h_1 c_0\,(\rho^7)$	$\tau h_1 \cdot h_1 h_3\,(\rho^2)$
			$h_0 h_3\,(\rho)$	$h_1^2 c_0\,(\rho^4)$	$\tau c_0\,(\rho^3)$
			$h_0^2 h_3\,(\rho)$	$h_1^{k+3} c_0\,(\rho^3)$	$h_1 \cdot \tau c_0\,(\rho^2)$

Table 8: Multiplicative generators of $\mathrm{Ext}_{\mathbb{R}}$

(s,f,w)	generator	ambiguity	definition
$(0,1,0)$	h_0	ρh_1	$\rho \cdot h_0 = 0$
$(1,1,1)$	h_1		
$(1,1,0)$	τh_1	$\rho^2 h_2$	$\rho^2 \cdot \tau h_1 = 0$
$(3,1,2)$	h_2		
$(0,1,-2)$	$\tau^2 h_0$		
$(3,1,0)$	$\tau^2 h_2$	$\rho^4 h_3$	$\rho^4 \cdot \tau^2 h_2 = 0$
$(6,2,3)$	τh_2^2	$\rho^2 h_1 h_3$	$\rho^6 \cdot \tau h_2^2 = 0$
$(7,1,4)$	h_3		
$(8,3,5)$	c_0	$\rho h_1^2 h_3$	$\rho^7 \cdot c_0 = 0$
$(0,1,-4)$	$\tau^4 h_0$		
$(8,3,4)$	τc_0		

Table 9: $\mathrm{Ext}_{\mathbb{R}}$ multiplication table

	h_0	h_1	τh_1	h_2	$\tau^2 h_0$
h_0	$-$				
h_1	0	$-$			
τh_1	$\rho h_1 \cdot \tau h_1$	$-$	$-$		
h_2	$-$	0	0	$-$	
$\tau^2 h_0$	$-$	$\rho(\tau h_1)^2$	$\rho^5 \tau h_2^2$	$h_0 \cdot \tau^2 h_2$	$\tau^4 h_0 \cdot h_0$
$\tau^2 h_2$	$-$	$\rho^2 \tau h_2^2$	$\rho^2 \tau^2 h_2 \cdot h_2$	$-$	
τh_2^2	0	ρc_0	$\rho \tau c_0$	$\tau h_1 \cdot h_1 h_3$	
h_3	$-$	$-$	$-$	0	
c_0	0	$-$	$h_1 \cdot \tau c_0$	0	
$\tau^4 h_0$	$-$	0			
τc_0	$\rho h_1 \cdot \tau c_0$	$-$			

Table 10: Some relations in $\mathrm{Ext}_{\mathbb{R}}$

(s,f,w)	relation
$(3,3,2)$	$h_0^2 h_2 + \tau h_1 \cdot h_1^2 = 0$
$(3,3,0)$	$h_0^2 \cdot \tau^2 h_2 + (\tau h_1)^3 = \rho^5 c_0$
$(9,3,6)$	$h_1^2 h_3 + h_2^3 = 0$
$(6,3,4)$	$h_0 h_2^2 = 0$
$(3,4,0)$	$h_0^3 \cdot \tau^2 h_2 = \rho^6 h_1 c_0$
$(7,5,4)$	$h_0^4 h_3 = \rho^3 h_1^2 c_0$
$(6,3,2)$	$h_0 h_2 \cdot \tau^2 h_2 = \rho^2 \cdot \tau c_0$
$(10,5,6)$	$h_1^2 \cdot \tau c_0 = 0$

Table 11: Notation for $\hat{\pi}_{*,*}$

(s,w)	element	Ext	definition
$(-1,-1)$	ρ	ρ	$\{\pm 1\} \to (\mathbb{A}^1 - 0)$
$(0,0)$	ϵ	1	twist on $S^{1,1} \wedge S^{1,1}$
$(0,0)$	ω	h_0	$1 - \epsilon$
$(1,1)$	η	h_1	Hopf construction [4] [13]
$(1,0)$	$\tau\eta$	τh_1	$\eta_{\text{top}} + \rho^2\nu$
$(3,2)$	ν	h_2	Hopf construction [4]
$(0,-2)$	$\tau^2\omega$	$\tau^2 h_0$	realizes to 2
$(3,0)$	$\tau^2\nu$	$\tau^2 h_2$	$\nu_{\text{top}} + \rho^4\sigma$
$(6,3)$	$\tau\nu^2$	τh_2^2	
$(7,4)$	σ	h_3	Hopf construction [4]
$(8,5)$	$\overline{\epsilon}$	c_0	$\rho\overline{\epsilon} = \eta \cdot \tau\nu^2$

10. Charts

This section contains the charts necessary to carry out the computations of the article.

Figure 1 depicts $\mathrm{Ext}_{\mathbb{C}}$ in a range. This data is lifted directly from [3] or [10]. Here is a key for reading Figure 1:

(1) Black dots indicate copies of $\mathbb{M}_2^{\mathbb{C}}$.
(2) Red dots indicate copies of $\mathbb{M}_2^{\mathbb{C}}/\tau$.
(3) Lines indicate multiplications by h_0, h_1, and h_2.
(4) Red arrows indicate infinitely many copies of $\mathbb{M}_2^{\mathbb{C}}/\tau$ that are connected by h_1 multiplications.
(5) Magenta lines indicate that a multiplication hits τ times a generator. For example, $h_0 \cdot h_0h_2$ equals τh_1^3.

Figure 2 depicts the various pages of the ρ-Bockstein spectral sequence, sorted by Milnor-Witt degree. See Section 5 for further discussion. Here is a key for reading Figure 2:

(1) Black dots indicate copies of $\mathbb{F}_2$.
(2) Red lines indicate multiplications by ρ.
(3) Green lines indicate multiplications by h_0.
(4) Blue lines indicate multiplications by h_1.
(5) Red (or blue) arrows indicate infinitely many copies of $\mathbb{F}_2$ that are connected by ρ (or η) multiplications.

Figure 3 depicts $\mathrm{Ext}_{\mathbb{R}} = \mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)$, sorted by Milnor-Witt degree. The key for this figure is the same as for Figure 2, with the additional:

(1) Dashed lines indicate h_0 or h_1 multiplications that are hidden in the ρ-Bockstein spectral sequence.

See Section 6 for discussion of these hidden extensions.

Figure 4 depicts the Milnor-Witt modules Π_0, Π_1, Π_2, and Π_3. See Section 8 for further discussion. Here is a key for reading the diagram:

(1) Open circles denote copies of $\mathbb{Z}_2$.
(2) Solid dots denote copies of $\mathbb{Z}/2$.
(3) Open circles with an n inside denote copies of $\mathbb{Z}/n$.
(4) Blue lines depict η-multiplications going to the right.
(5) Red lines depict ρ-multiplications going to the left.
(6) Lines labelled n indicate that the result of the multiplication is n times the labelled generator: for example, $\rho \cdot \rho\eta = -2\rho$ or $\eta \cdot \rho^3 = -2\rho^2$ in Π_0.
(7) Two blue (or red lines) with the same source indicate that the multiplication by η (or ρ) equals a linear combination. For example, $\eta \cdot \tau\nu^2$ equals $(\rho\bar{\epsilon} - 4\sigma) + 4\sigma = \rho\bar{\epsilon}$ in Π_3.
(8) Arrows pointing off the diagram indicate infinitely many multiplications by ρ or by η.
(9) Elements in the same topolgical stem are aligned vertically. For example, η^3, ν, $\eta(\tau\eta)^2$, and $\rho^3\nu^2$ all belong to the 3-stem. Their weights are 1, 2, 1, and 1 respectively; this can be deduced from their stems and Milnor-Witt degrees.

FIGURE 1: $\mathrm{Ext}_{\mathbb{C}} = \mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})$

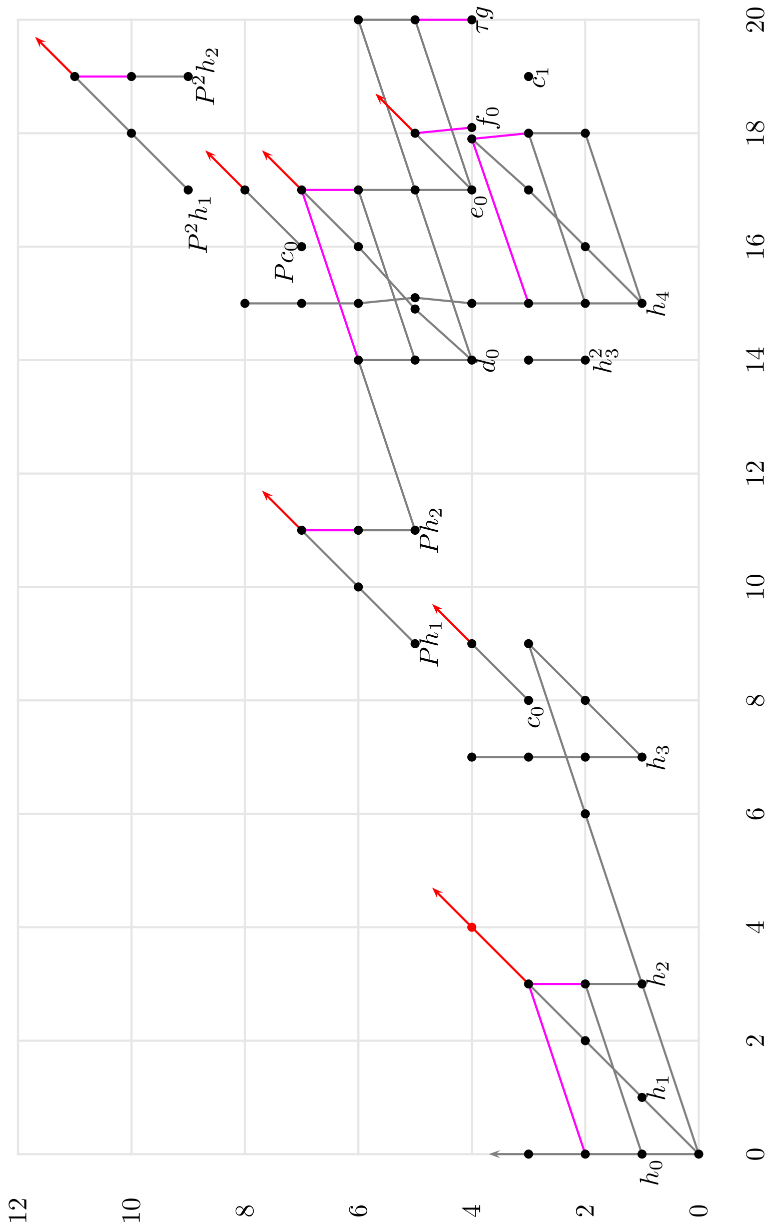
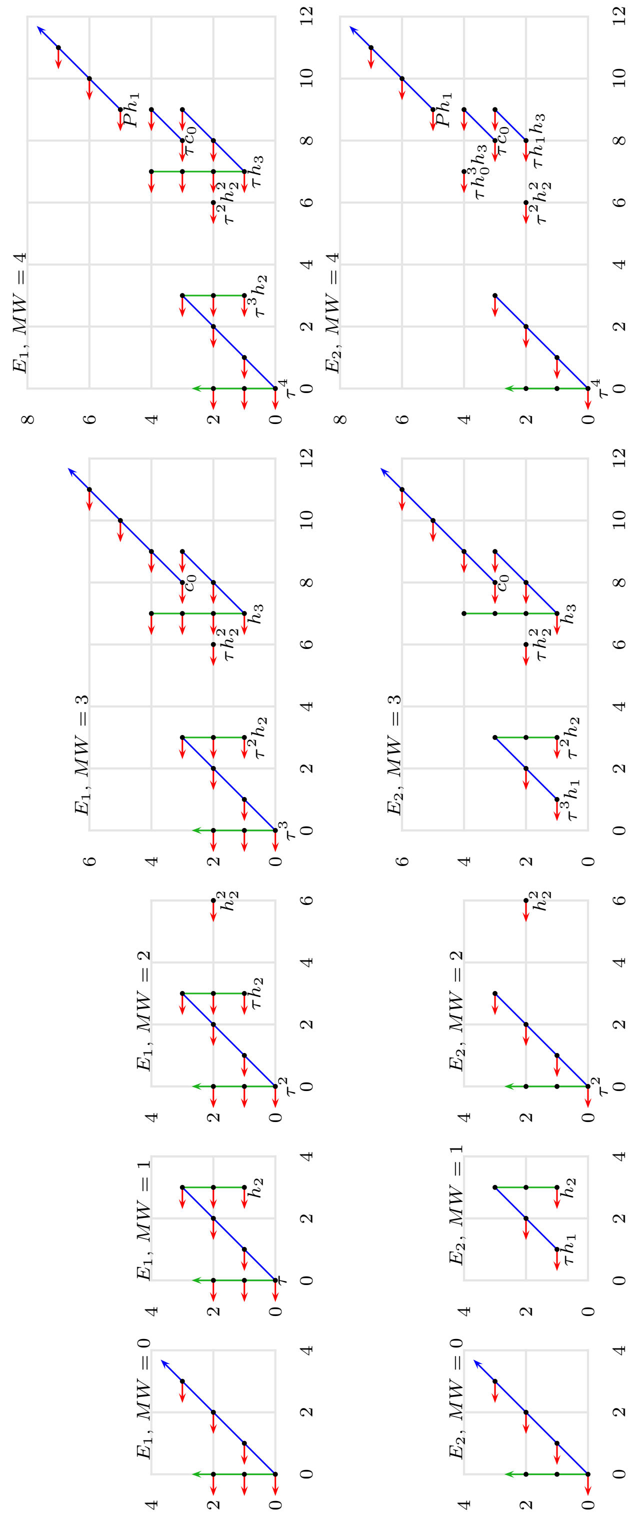


FIGURE 2: ρ-Bockstein spectral sequence



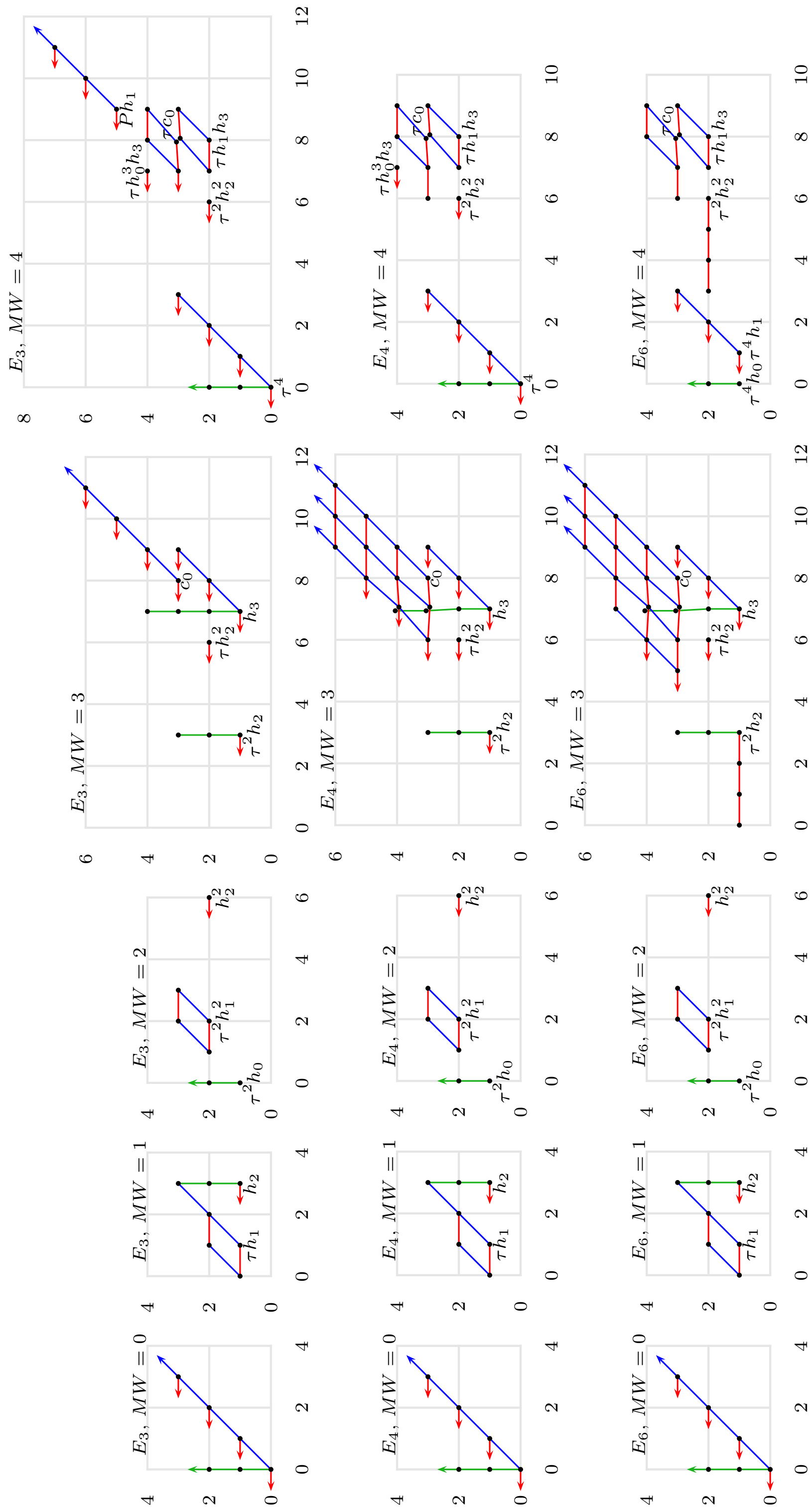


FIGURE 2: ρ-Bockstein spectral sequence (continued)

FIGURE 2: ρ-Bockstein spectral sequence (continued)

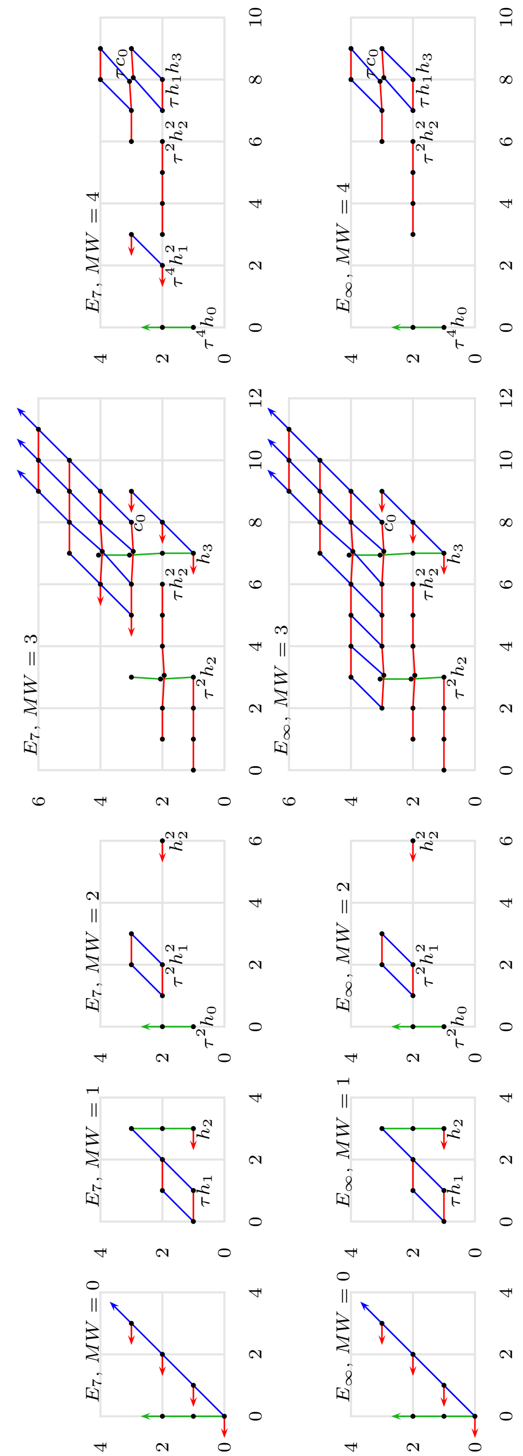


FIGURE 3: $\mathrm{Ext}_{\mathbb{R}} = \mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)$

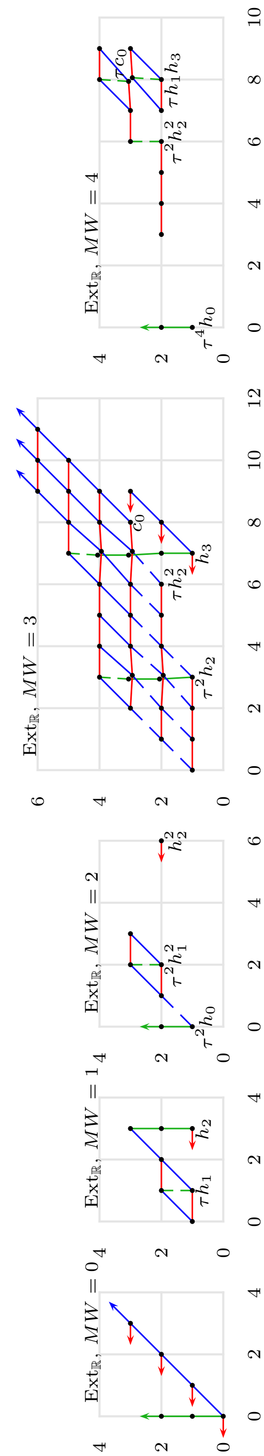
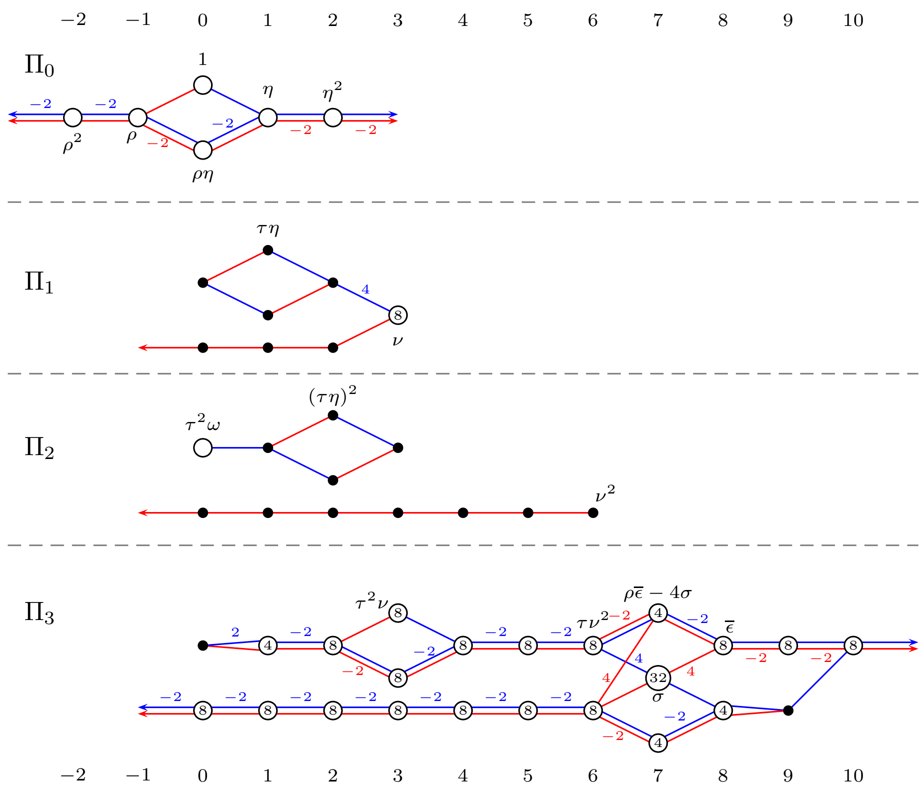


FIGURE 4: Milnor-Witt modules



REFERENCES

[1] Shôrô Araki and Kouyemon Iriye, *Equivariant stable homotopy groups of spheres with involutions. I*, Osaka J. Math. **19** (1982), no. 1, 1–55. MR656233 (83h:55028)

[2] Simone Borghesi, *Cohomology operations and algebraic geometry*, Proceedings of the Nishida Fest (Kinosaki 2003), Geom. Topol. Monogr., vol. 10, Geom. Topol. Publ., Coventry, 2007, pp. 75–115, DOI 10.2140/gtm.2007.10.75, (to appear in print). MR2402778 (2009m:14024)

[3] Daniel Dugger and Daniel C. Isaksen, *The motivic Adams spectral sequence*, Geom. Topol. **14** (2010), no. 2, 967–1014, DOI 10.2140/gt.2010.14.967. MR2629898 (2011e:55024)

[4] ———, *Motivic Hopf elements and relations*, New York J. Math. **19** (2013), 823–871. MR3141814

[5] Bertrand J. Guillou and Daniel C. Isaksen, *The motivic vanishing line of slope* 1/2, in preparation.

[6] Michael A. Hill, *Ext and the motivic Steenrod algebra over* ℝ, J. Pure Appl. Algebra **215** (2011), no. 5, 715–727, DOI 10.1016/j.jpaa.2010.06.017. MR2747214 (2012i:55020)

[7] Mark Hovey, *Morita theory for Hopf algebroids and presheaves of groupoids*, Amer. J. Math. **124** (2002), no. 6, 1289–1318. MR1939787 (2003k:16053)

[8] P. Hu, I. Kriz, and K. Ormsby, *Convergence of the motivic Adams spectral sequence*, J. K-Theory **7** (2011), no. 3, 573–596, DOI 10.1017/is011003012jkt150. MR2811716 (2012h:14054)

[9] Daniel C. Isaksen, *Stable stems* (2014), preprint, available at `arXiv:1407.8418`.

[10] ———, *Classical and motivic Adams charts* (2014), available at `arXiv:1401.4983`.

[11] J. Peter May, *Matric Massey products*, J. Algebra **12** (1969), 533–568. MR0238929 (39 #289)

[12] ———, *A general algebraic approach to Steenrod operations*, The Steenrod Algebra and its Applications (Proc. Conf. to Celebrate N. E. Steenrod's Sixtieth Birthday, Battelle Memorial Inst., Columbus, Ohio, 1970), Lecture Notes in Mathematics, Vol. 168, Springer, Berlin, 1970, pp. 153–231. MR0281196 (43 #6915)

[13] Fabien Morel, *An introduction to $\mathbb{A}^1$-homotopy theory*, Contemporary developments in algebraic K-theory, ICTP Lect. Notes, XV, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004, pp. 357–441 (electronic). MR2175638 (2006m:19007)
[14] ———, *On the motivic π_0 of the sphere spectrum*, Axiomatic, enriched and motivic homotopy theory, NATO Sci. Ser. II Math. Phys. Chem., vol. 131, Kluwer Acad. Publ., Dordrecht, 2004, pp. 219–260, DOI 10.1007/978-94-007-0948-5, (to appear in print). MR2061856 (2005e:19002)
[15] ———, *The stable $\mathbb{A}^1$-connectivity theorems*, K-Theory **35** (2005), no. 1-2, 1–68, DOI 10.1007/s10977-005-1562-7. MR2240215 (2007d:14041)
[16] R. Michael F. Moss, *Secondary compositions and the Adams spectral sequence*, Math. Z. **115** (1970), 283–310. MR0266216 (42 #1123)
[17] Kyle M. Ormsby and Paul Arne Østvær, *Stable motivic π_1 of low-dimensional fields*, Adv. Math. **265** (2014), 97–131, DOI 10.1016/j.aim.2014.07.024. MR3255457
[18] Hirosi Toda, *Composition methods in homotopy groups of spheres*, Annals of Mathematics Studies, No. 49, Princeton University Press, Princeton, N.J., 1962. MR0143217 (26 #777)
[19] Glen Wilson and Knight Fu (2014), personal communication.
[20] Vladimir Voevodsky, *Reduced power operations in motivic cohomology*, Publ. Math. Inst. Hautes Études Sci. **98** (2003), 1–57, DOI 10.1007/s10240-003-0009-z. MR2031198 (2005b:14038a)

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