Area and the Method of Exhaustion

The Greeks assumed on intuitive grounds that simple curvilinear figures, such as circles or ellipses, have areas that are geometric magnitudes of the same type as areas of polygonal figures, and that these areas enjoy the following two natural properties.

(i) (Monotonicity) If S is contained in T, then $a(S) \leqslant a(T)$.
(ii) (Additivity) If R is the union of the non-overlapping figures S and T, then $a(R) = a(S) + a(T)$.

Given a curvilinear figure S, they attempted to determine its area $a(S)$ by means of a sequence $P_1, P_2, P_3, \ldots,$ of polygons that fill up or "exhaust" S, analogous to Hippocrates' sequence of regular polygons inscribed in a circle. The so-called method of exhaustion was devised, apparently by Eudoxus, to provide a rigorous alternative to merely taking a vague and unexplained limit of $a(P_n)$ as $n \to \infty$. Indeed, the Greeks studiously avoided "taking the limit" explicitly, and this virtual "horror of the infinite" is probably responsible for the logical clarity of the method of exhaustion.

In any event, the crux of the matter consists of showing that the area $a(S - P_n)$, of the difference between the figure S and the inscribed polygon P_n, can be made as small as desired by choosing n sufficiently large. For this purpose the following consequence (Euclid X.1) of the Archimedes-Eudoxus axiom is repeatedly applied.

> Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out.

This result, which we will call "Eudoxus' principle," may be phrased as follows. Let M_0 and ϵ be the two given magnitudes, and $M_1, M_2, M_3, \ldots,$ a sequence such that $M_1 < \frac{1}{2}M_0$, $M_2 < \frac{1}{2}M_1$, $M_3 < \frac{1}{2}M_2$, etc. Then we want to conclude that $M_n < \epsilon$ for some n. To see that this is so, choose an integer N such that

$$(N+1)\epsilon > M_0.$$

Then ϵ is at most half of $(N+1)\epsilon$, so it follows that

$$N\epsilon \geqslant \tfrac{1}{2}M_0 > M_1.$$

Similarly, ϵ is at most half of $N\epsilon$, so

$$(N-1)\epsilon \geqslant \tfrac{1}{2}M_1 > M_2.$$

Proceeding in this way, at each step subtracting ϵ (which is at most half) from the left-hand-side and halving the right-hand-side, we arrive in N steps at the desired inequality

$$\epsilon > M_N. \qquad \square$$

Exercise 17. Conclude from Eudoxus' principle that, if M, ϵ, and $r < \frac{1}{2}$ are given positive numbers, then $Mr^n < \epsilon$ for n sufficiently large. Is it necessary that r be at most $\frac{1}{2}$?

We first apply Eudoxus' principle to describe precisely the manner in which the area of a circle can be exhausted by means of inscribed polygons.

Given a circle C and a number $\epsilon > 0$, there exists a regular polygon P inscribed in C such that

$$a(C) - a(P) < \epsilon. \tag{8}$$

Proof. We start with a square $P_0 = EFGH$ inscribed in the circle C, and write $M_0 = a(C) - a(P_0)$. Doubling the number of sides, we obtain a regular octagon P_1 inscribed in C (Fig. 15).

Continuing in this fashion, we obtain a sequence $P_0, P_1, P_2, \ldots, P_n, \ldots,$ with P_n having 2^{n+2} sides. Writing

$$M_n = a(C) - a(P_n),$$

we want to show that

$$M_n - M_{n+1} > \tfrac{1}{2}M_n. \tag{9}$$

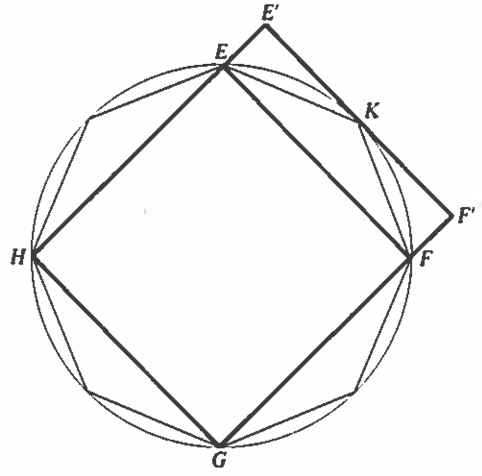

Figure 15

It will then follow from Eudoxus' principle that $M_n < \epsilon$ for n sufficiently large, and we will be finished.

The proof of (9) is essentially the same for all n, so we consider the case $n=0$ illustrated by Fig. 15. Then

$$\begin{aligned} M_0 - M_1 &= a(P_1) - a(P_0) \\ &= 4a(\Delta EFK) \\ &= 2a(EFF'E') \\ &> 2a(\widehat{EKF}) \\ &= \tfrac{1}{2}\cdot 4a(\widehat{EKF}) \\ &= \tfrac{1}{2}[a(C) - a(P_0)] \\ M_0 - M_1 &> \tfrac{1}{2}M_0, \end{aligned}$$

where we denote by $\widehat{EFK}$ the circular segment cut off the circle by the side EF of the square P_0. In the general case, we obtain

$$\begin{aligned} M_n - M_{n+1} &= a(P_{n+1}) - a(P_n) \\ &> \tfrac{1}{2}[a(C) - a(P_n)] = \tfrac{1}{2}M_n, \end{aligned}$$

where $a(C) - a(P_n)$ is the sum of the areas of the 2^{n+1} circular segments cut off by the edges of P_n. □

The above lemma provides the basis for a rigorous proof of the theorem on areas of circles (Euclid XII.2).

If C_1 and C_2 are circles with radii r_1 and r_2, then

$$\frac{a(C_1)}{a(C_2)} = \frac{r_1^2}{r_2^2}. \tag{10}$$

Proof. If $A_1 = a(C_1)$, $A_2 = a(C_2)$, then either

$$\frac{A_1}{A_2} = \frac{r_1^2}{r_2^2} \quad \text{or} \quad \frac{A_1}{A_2} < \frac{r_1^2}{r_2^2} \quad \text{or} \quad \frac{A_1}{A_2} > \frac{r_1^2}{r_2^2}.$$

The proof is a double *reductio ad absurdum* argument, characteristic of Greek geometry, in which we show that the assumption of either of the inequalities leads to a contradiction.

Suppose first that

$$\frac{A_1}{A_2} < \frac{r_1^2}{r_2^2}, \quad \text{or} \quad A_2 > \frac{A_1 r_2^2}{r_1^2} = S,$$

and let $\epsilon = A_2 - S > 0$. Then, by the lemma, there exists a polygon P_2 inscribed in C_2 such that

$$A_2 - a(P_2) < \epsilon = A_2 - S,$$

so $a(P_2) > S$. But

$$\frac{a(P_1)}{a(P_2)} = \frac{r_1^2}{r_2^2} = \frac{A_1}{S}, \quad \text{(Exercise 9)}$$

where P_1 is the similar regular polygon inscribed in C_1. It follows that

$$\frac{S}{a(P_2)} = \frac{A_1}{a(P_1)} = \frac{a(C_1)}{a(P_1)} > 1$$

so $S > a(P_2)$, which is a contradiction. Hence the assumption $A_1/A_2 < r_1^2/r_2^2$ is false.

By interchanging the roles of the two circles, we find similarly that the assumption

$$\frac{A_1}{A_2} > \frac{r_1^2}{r_2^2} \quad \text{or} \quad \frac{A_2}{A_1} < \frac{r_2^2}{r_1^2}$$

is also false. We therefore conclude that (10) holds, as desired. □

If we rewrite (10) as

$$\frac{a(C_1)}{r_1^2} = \frac{a(C_2)}{r_2^2}, \tag{11}$$

and denote by π the common value of these two ratios, then we obtain the familiar formula $A = \pi r^2$ for the area of a circle. In fact, however, the Greeks could not do this, because for them (11) was a proportion between ratios of areas, rather than a numerical equality. Hence the number π does not appear in this connection in Greek mathematics.

Exercise 18. Apply the lemma on the exhaustion of a circle by inscribed polygons, together with the fact that the volume of a prism is the product of its height and the area of its base, to give a double *reductio ad absurdum* proof that the volume of a circular cylinder is equal to the product of its height and the area of its base. Given a polygon P inscribed in the base circle, consider the prism Q with base P and height equal to that of the cylinder. Then the cylinder can be exhausted by prisms like Q.

Volumes of Cones and Pyramids

If P is either a triangular pyramid or a circular cone, then its volume is given by

$$v(P) = \tfrac{1}{3}Ah, \tag{12}$$

where h is its height and A the area of its base. According to Archimedes, the two results described by this formula were discovered by Democritus,

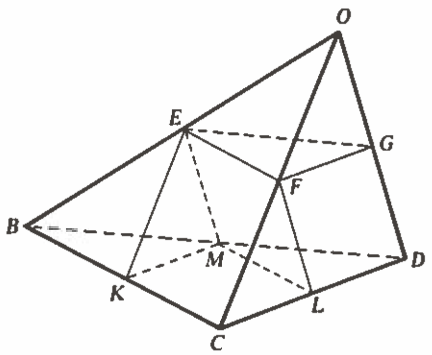

Figure 16

but were first proved by Eudoxus. In this section we discuss their treatment by Euclid in Book XII of the *Elements*.

The calculation of the volume of a pyramid is based on the dissection of an arbitrary pyramid with triangular base into two prisms and two similar pyramids, as indicated in Figure 16. The points E, F, G, K, L, M are the midpoints of the six edges of the pyramid $OBCD$. It is clear that the pyramids $OEFG$ and $EBKM$ are similar to $OBCD$ and are congruent to each other. The crucial fact about this dissection is that the sum of the volumes of the two prisms

$$EKMFCL \quad \text{and} \quad MLDEFG$$

is *greater than half* the volume of the original pyramid $OBCD$. This is true because

$$v(OEFG) = v(FKCL) < v(EKMFCL)$$

and

$$v(EBKM) = v(GMLD) < v(MLDEFG).$$

If we denote by h the height and by A the area of the base BCD of the pyramid $OBCD$, then

$$v(MLDEFG) = \tfrac{1}{8}Ah$$

because the height of this prism is $\frac{1}{2}h$ and the area of its base MLD is $\frac{1}{4}A$. Also,

$$v(EKMFCL) = \tfrac{1}{8}Ah$$

because the area of the parallelogram $KCLM$ is $\frac{1}{2}A$, and the prism $EKMFCL$ is half of a parallelepiped with base $KCLM$ and height $\frac{1}{2}h$. Consequently the sum of the volumes of the two smaller prisms is $\frac{1}{4}Ah$.

Now let us similarly dissect each of the two pyramids $OEFG$ and $EBKM$ into two smaller pyramids and two prisms. The sum of the volumes of the

four resulting smaller prisms is then greater than half of the sum of the volumes of the pyramids $OEFG$ and $EBKM$. Because these two latter pyramids both have height $h/2$ and base area $A/4$, it follows that the sum of the volumes of the four smaller prisms is

$$2 \cdot \frac{1}{4} \cdot \frac{A}{4} \cdot \frac{h}{2} = \frac{Ah}{4^2}.$$

After n steps of this sort, we obtain an *n-step-dissection* of the original pyramid. At the kth step we have 2^k subdivided small pyramids, and hence 2^k pairs of smaller prisms. Each of the 2^k small pyramids has height $h/2^k$ and base area $A/4^k$, so the sum of the volumes of the 2^k pairs of smaller prisms is

$$2^k \cdot \frac{1}{4} \cdot \frac{A}{4^k} \cdot \frac{h}{2^k} = \frac{Ah}{4^{k+1}}.$$

Finally, if P denotes the union of all the prisms obtained in all the steps of this n-step-dissection, it follows that

$$v(P) = Ah\left(\frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^{n+1}}\right). \tag{13}$$

Furthermore, because at each step the sum of the volumes of the prisms is greater than half the sum of the pyramids obtained in the previous step, Eudoxus' principle implies that, given $\epsilon > 0$,

$$V - v(P) < \epsilon \tag{14}$$

if n is sufficiently large, and $V = v(OBCD)$. This construction is the basis for Euclid's proof of Proposition XII.5.

Given two triangular pyramids with the same height and with base areas A_1 and A_2, the ratio of their volumes V_1 and V_2 is equal to that of their base areas,

$$\frac{V_1}{V_2} = \frac{A_1}{A_2}. \tag{15}$$

Proof. The demonstration of (15) is a double *reductio ad absurdum* argument almost identical to that used in the proof of the theorem on areas of circles. Suppose first that

$$\frac{V_1}{V_2} < \frac{A_1}{A_2}, \quad \text{or} \quad V_2 > \frac{V_1 A_2}{A_1} = S,$$

and let $\epsilon = V_2 - S$. Denote by P_2 the union of all the prisms obtained in an n-step-dissection of the second pyramid, with n sufficiently large that

$$V_2 - v(P_2) < \epsilon = V_2 - S,$$

so $v(P_2)>S$. It then follows from (13) that, if P_1 is the similar union of prisms obtained in an n-step-dissection of the first pyramid, then

$$\frac{v(P_1)}{v(P_2)}=\frac{A_1}{A_2}=\frac{V_1}{S}.$$

Hence

$$\frac{S}{v(P_2)}=\frac{V_1}{v(P_1)}>1$$

because P_1 is properly contained in the first pyramid. But $S>v(P_2)$ is a contradiction, so the assumption $V_1/V_2<A_1/A_2$ is false.

By interchanging the roles of the two pyramids, we find that the assumption $V_1/V_2>A_1/A_2$ is also false. It therefore follows that $V_1/V_2=A_1/A_2$, as desired. □

We have previously seen that the formula $V=\frac{1}{3}Ah$, for the volume of a triangular pyramid, follows from the fact that two pyramids with equal heights and base areas must have the same volumes. For any given pyramid is one of three pyramids with equal volumes, whose union is a prism with height and base area equal to those of the given pyramid (see Figure 10). We assume here (as in the above construction) the elementary fact that the volume of a prism is the product of its height and its base area.

Alternatively, it is interesting to derive the formula $V=\frac{1}{3}Ah$ directly by using the sum of the geometric series

$$\sum_{n=0}^{\infty}\frac{1}{4^n}=\frac{4}{3}.$$

Given $\epsilon>0$, we see from (13) that

$$V-Ah\left(\frac{1}{4}+\frac{1}{4^2}+\cdots+\frac{1}{4^{n+1}}\right)<\epsilon$$

if n is sufficiently large. It follows that the volume of the pyramid is

$$V=\frac{Ah}{4}\sum_{n=0}^{\infty}\frac{1}{4^n}=\frac{Ah}{4}\cdot\frac{4}{3}=\frac{1}{3}Ah.$$

Although the Greeks knew how to sum a finite geometric progression, they used *reductio ad absurdum* arguments to avoid the formal summation of an infinite series.

EXERCISE 19. Show that the volume formula $V=\frac{1}{3}Ah$ holds for a pyramid whose base is an arbitrary convex polygon (that can be dissected into triangles, thereby dissecting the pyramid into triangular pyramids for which the formula is already known).

EXERCISE 20. Show that the ratio of the volumes of two similar pyramids is equal to the ratio of the cubes of corresponding edges.

In Proposition XII.10 Euclid uses inscribed pyramids to exhaust a circular cone so as to establish the volume formula $V=\frac{1}{3}Ah$ for cones. To outline this proof, let T be a cone with vertex O, base circle C, and height h. Let

$$P_0, P_1, P_2, \ldots, P_n, \ldots$$

be the sequence of inscribed regular polygons previously used to exhaust the circle, with P_n having 2^{n+2} sides. If T_n denotes the pyramid with vertex O and base P_n, then

$$T_0, T_1, T_2, \ldots, T_n, \ldots$$

is a sequence of pyramids inscribed in the cone T, and $v(T_n)=\frac{1}{3}a(P_n)h$, where h is the height of T.

Recall that we proved that, if $M_n=a(C)-a(P_n)$, then $M_n-M_{n+1}>\frac{1}{2}M_n$. By joining every polygon involved with the vertex O, we can similarly prove that, if

$$\overline{M_n}=v(T)-v(T_n),$$

then

$$\overline{M_n}-\overline{M_{n+1}}>\tfrac{1}{2}\overline{M_n}.$$

Eudoxus' principle therefore implies that, given $\epsilon>0$,

$$\overline{M_n}=v(T)-v(T_n)<\epsilon \tag{16}$$

if n is sufficiently large. Also, if Q is the cylinder with base C and height h, and Q_n is the inscribed prism with base P_n and height h, then

$$v(Q)-v(Q_n)<\epsilon$$

for n sufficiently large (why?).

We are now ready for the *reductio ad absurdum* proof that

$$v(T)=\tfrac{1}{3}v(Q)=\tfrac{1}{3}Ah. \tag{17}$$

Otherwise, either $v(T)<\frac{1}{3}v(Q)$ or $v(T)>\frac{1}{3}v(Q)$.

Assuming that $v(T)<\frac{1}{3}v(Q)$, choose n sufficiently large that

$$v(Q)-v(Q_n)<v(Q)-3v(T).$$

Then $v(Q_n)>3v(T)>3v(T_n)$ because the pyramid T_n is inscribed in the cone T. But the conclusion that $v(T_n)<\frac{1}{3}v(Q_n)$ is a contradiction, because the pyramid T_n and the prism Q_n have the same base and height, so we know (Exercise 19) that $v(T_n)=\frac{1}{3}v(Q_n)$.

Assuming that $v(T) > \frac{1}{3}v(Q)$, choose n sufficiently large that

$$v(T) - v(T_n) < v(T) - \tfrac{1}{3}v(Q).$$

Then $v(T_n) > \frac{1}{3}v(Q) > \frac{1}{3}v(Q_n)$ because the prism Q_n is inscribed in the cylinder Q. But this is a contradiction for the same reason as before, so we conclude that $v(T) = \frac{1}{3}v(Q)$ as desired. □

Volumes of Spheres

The final result in Book XII of the *Elements* is Proposition 18, to the effect that the volume of a sphere is proportional to the cube of its radius. Euclid proves this in the following form.

If S_1 and S_2 are two spheres with radii r_1 and r_2 and volumes V_1 and V_2, then

$$\frac{V_1}{V_2} = \frac{r_1^3}{r_2^3}. \tag{18}$$

As a preliminary lemma (XII.17) he shows that, given two concentric spheres S and S' with S' interior to S, there exists a polyhedral solid P inscribed in S that contains S' in its interior. The polyhedral solid P is a union of finitely many pyramids, each of which has the common center O of the two spheres as its vertex, with its base being a polygon inscribed in the outer sphere S (Fig. 17).

In his proof of Proposition 18, Euclid assumes without proof that, given a sphere S with volume V and $V' < V$, there exists a concentric sphere S' with $v(S') = V'$. We will repair this minor gap by using the slightly simpler

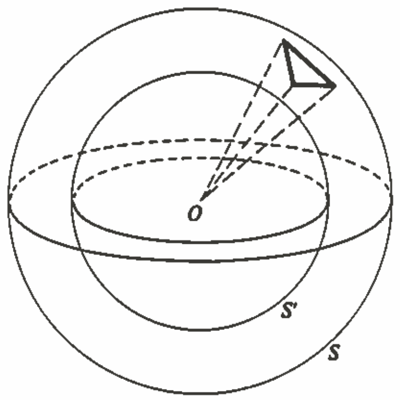

Figure 17

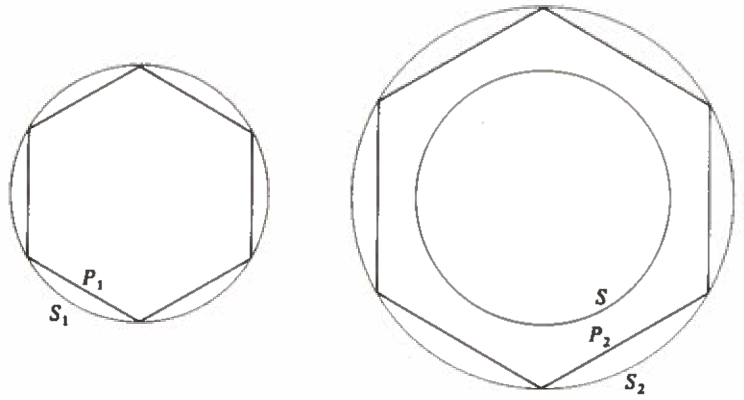

Figure 18

fact that there exists a concentric sphere S' with $V' < v(S') < V$ (Exercise 21 below).

Assuming that $V_1/V_2 < r_1^3/r_2^3$, let

$$\epsilon = V_2 - \frac{r_2^3 V_1}{r_1^3},$$

and let S be a sphere interior to and concentric with S_2 (see Fig. 18) such that

$$v(S) = V > V_2 - \epsilon = \frac{r_2^3 V_1}{r_1^3}. \tag{19}$$

Now let P_2 be a polyhedral solid inscribed in S_2 that contains S in its interior. If P_1 is the similar polyhedral solid inscribed in S_1, then

$$\frac{v(P_1)}{v(P_2)} = \frac{V_1'}{V_2'} = \frac{r_1^3}{r_2^3} \tag{20}$$

by Exercise 20, because P_1 and P_2 are made up of pairwise similar pyramids with corresponding edges r_1 and r_2. Hence

$$V_2' > V > \frac{r_2^3 V_1}{r_1^3}$$

by (19), so

$$\frac{V_1}{V_2'} < \frac{r_1^3}{r_2^3} = \frac{V_1'}{V_2'}$$

by (20). Thus $v(S_1) = V_1 < V_1' = v(P_1)$. But this is a contradiction, because S_1 contains P_1.

Interchanging the roles of the two spheres S_1 and S_2, the assumption that $V_1/V_2 > r_1^3/r_2^3$ leads similarly to a contradiction. Consequently we conclude that $V_1/V_2 = r_1^3/r_2^3$, as desired. □