LARGE ANNIHILATORS IN CAYLEY-DICKSON ALGEBRAS II

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Abstract. We establish many previously unknown properties of zero-divisors in Cayley-Dickson algebras. The basic approach is to use a certain splitting that simplifies computations surprisingly.

1. Introduction

Cayley-Dickson algebras are non-associative finite-dimensional $\mathbb{R}$-division algebras that generalize the real numbers, the complex numbers, the quaternions, and the octonions. This paper is a sequel to [DDD], which explores some detailed algebraic properties of these algebras.

Classically, the first four Cayley-Dickson algebras, i.e., $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, are viewed as at least somewhat well-behaved, while the larger Cayley-Dickson algebras are considered to be pathological. There are several different ways of making this distinction. One difference is that the first four algebras do not possess zero-divisors, while the higher algebras do have zero-divisors. Our primary long-term goal is to understand the zero-divisors in as much detail as possible. The specific purpose of this paper is to build directly on the ideas of [DDD] about zero-divisors with large annihilators.

Our motivation for studying zero-divisors is their potential for useful applications in topology; see [Co] for more details. Also, [A] uses Cayley-Dickson algebras to construct new bilinear normed maps. Another significant reference is [ES], which computes the automorphism groups of all Cayley-Dickson algebras.

Let A_n be the Cayley-Dickson algebra of dimension 2^n. The central idea of the paper is to use a certain additive splitting of A_n (as expressed indirectly in Definition 3.1) to simplify multiplication formulas. Multiplication does not quite respect the splitting, but it almost does (see Proposition 4.1). Theorem 4.5 is the technical heart of the paper; it supplies expressions for multiplication of elements of a codimension 4 subspace of A_n that are simpler than one might expect.

These simple multiplication formulas lead to detailed information about zero-divisors and their annihilators. Section 5 takes a straightforward approach: just write out equations and solve them as explicitly as possible. Our simple multiplication formulas make this feasible. This leads to Theorem 5.10, which almost completely computes the dimension of the annihilator of any element. There are two ways in which the theorem fails to be complete. First, it only treats annihilators of elements in a codimension 4 subspace of A_n. Second, rather than determining the dimension of an annihilator precisely, it gives two options, which differ by 4.

This paper is dedicated to the memory of Guillermo Moreno, who made many contributions to the study of Cayley-Dickson algebras.

We currently have no solution to the first problem. However, in this regard, it was already known that one codimension 2 slice is easy to deal with, so the restriction is really only codimension 2. We intend to address this question in future work.

The second problem has a partial solution in Theorems 6.7 and 6.12, which distinguish between the two possible cases. We find that the answer for A_{n+1} depends inductively not just on an understanding of zero-divisors in A_n but also on a detailed understanding of annihilators in A_n (see Definition 6.1). Therefore, the description in these theorems is not as explicit as we might like.

Fortunately, we have a complete understanding of zero-divisors and their annihilators in A_4 [KY, Section 3.2] [M1, Corollary 2.14] [DDD, Sections 11 and 12]. This allows us to make calculations about zero-divisors in A_5 that are not yet possible for A_n with $n \geq 6$. Section 7 contains the details of these calculations in A_5. Consequently, even though we have not made this result explicit in this article, it is possible to completely understand in geometric terms the zero-divisors in a codimension 4 subspace of A_5. This goes a long way towards completely describing the zero-divisors of A_5.

In addition to the concrete results in Section 7 about A_5, Section 8 gives a number of results about spaces of zero-divisors in A_n for arbitrary n. Consider for a moment only the zero-divisors whose annihilators have dimension differing from the maximum possible dimension by a fixed constant. We show in Theorem 8.12 that, in a certain sense, the space of such zero-divisors does not depend on n. This is a kind of stability result for zero-divisors with large annihilators; it was alluded to in [DDD, Remark 15.8]. The basic approach is to use the previous calculations of dimensions of annihilators, together with bounds on the dimensions of annihilators from [DDD] (see Theorem 2.11).

The paper contains a review in Section 2 of the key properties of Cayley-Dickson algebras that we will use. Only some of the material is original; it quotes many results from [DDD] that will be relevant here.

We make one further remark about generalities. Many of our results have hypotheses that eliminate consideration of the classical algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, even though sometimes this is not strictly necessary. From the perspective of this paper, these low-dimensional algebras behave significantly differently than A_n for $n \geq 4$. We eliminate them to avoid awkward but easy special cases.

1.1. **Statement of Results.** We now present a summary of our technical results.

Recall that A_n is additively isomorphic to $A_{n-1} \times A_{n-1}$, so elements of A_n are expressions (a, b), where a and b belong to A_{n-1}. See Section 2 for a multiplication formula with respect to this notation.

The element $i_n = (0, 1)$ of A_n has many special properties that will be described below. Let $\mathbb{C}_n$ be the linear span of $1 = (1, 0)$ and i_n; it is a subalgebra of A_n isomorphic to the complex numbers. Let $\mathbb{H}_{n+1}$ be the linear span of $(1, 0)$, $(0, 1)$, $(i_n, 0)$, and $(0, i_n)$; it is a subalgebra of A_{n+1} isomorphic to the quaternions.

It turns out that A_n is naturally a Hermitian inner product space. The Hermitian inner product of two elements a and b is the orthogonal projection of ab^* onto $\mathbb{C}_n$. We say that two elements a and b are $\mathbb{C}$-orthogonal if their Hermitian inner product is zero.

Results of [DDD] suggest that we should pay particular attention to elements of A_{n+1} of the form $(a, \pm i_n a)$ with a in the orthogonal complement $\mathbb{C}_n^{\perp}$ of $\mathbb{C}_n$. Every element of the orthogonal complement $\mathbb{H}_{n+1}^{\perp}$ of $\mathbb{H}_{n+1}$ can be written uniquely in the form

$$\frac{1}{\sqrt{2}}\Big(a, -i_n a\Big) + \frac{1}{\sqrt{2}}\Big(b, i_n b\Big),$$

where a and b belong to $\mathbb{C}_n^{\perp}$. We use the notation $\{a, b\}$ for this expression. We insert the ungainly scalars $\frac{1}{\sqrt{2}}$ in order to properly normalize some formulas that appear later. We would like to consider the product of two elements $\{a, b\}$ and $\{x, y\}$ of $\mathbb{H}_{n+1}^{\perp}$.

Proposition 1.2. *Let a, b, x, and y belong to $\mathbb{C}_n^{\perp}$, and suppose that a and b are $\mathbb{C}$-orthogonal to both x and y. Then*

$$\{a, b\}\{x, y\} = \sqrt{2}\{ax, by\}.$$

This result is proved at the beginning of Section 4. The formula is remarkably simple, but it is not completely general because of the orthogonality assumptions on a, b, x, and y. Most of Section 4 is dedicated to generalizing this formula and understanding the resulting error terms.

Recall that the annihilator $\mathrm{Ann}(x)$ of an element x of A_n is the set of all elements y such that $xy = 0$. Proposition 1.2 is the key computational step in the following theorem about annihilators, which is proved in Section 5.

Theorem 1.3. *Let $n \geq 3$, and let a and b be non-zero elements of $\mathbb{C}_n^{\perp}$. Then the dimension of the annihilator of $\{a, b\}$ is equal to $\dim \mathrm{Ann}\, a + \dim \mathrm{Ann}\, b$ or $\dim \mathrm{Ann}\, a + \dim \mathrm{Ann}\, b + 4$.*

In order to distinguish between the two cases of Theorem 1.3, we need the following definition.

Definition 1.4. *The D-locus is the space of all elements $\{a, b\}$ of A_{n+1} with a and b in $\mathbb{C}_n^{\perp}$ such that*

(1) a and b are $\mathbb{C}$-orthogonal,
(2) a and $\mathrm{Ann}(b)$ are orthogonal, and
(3) b and $\mathrm{Ann}(a)$ are orthogonal.

The following result is proved in Section 6.

Theorem 1.5. *Let a and b be non-zero elements of $\mathbb{C}_n^{\perp}$. If $\{a, b\}$ does not belong to the D-locus in A_{n+1}, then the dimension of the annihilator of $\{a, b\}$ is $\dim \mathrm{Ann}\, a + \dim \mathrm{Ann}\, b$. If $\{a, b\}$ belongs to the D-locus in A_{n+1}, then the dimension of the annihilator of $\{a, b\}$ is $\dim \mathrm{Ann}\, a + \dim \mathrm{Ann}\, b + 4$.*

For example, if neither a nor b are zero-divisors in A_n and a and b are not $\mathbb{C}$-orthogonal, then $\{a, b\}$ is not a zero-divisor. If neither a nor b are zero-divisors in A_n but are $\mathbb{C}$-orthogonal, then $\{a, b\}$ does belong to the D-locus in A_{n+1} and is thus a zero-divisor. On the other hand, the theorem also shows that if a or b is a zero-divisor, then $\{a, b\}$ is a zero-divisor regardless of whether or not it belongs to the D-locus. In summary, if a and b are $\mathbb{C}$-orthogonal, then $\{a, b\}$ is always a zero-divisor.

In Section 7, we explicitly work out the meaning of Definition 1.4 when a and b belong to A_4. The only difficult case occurs when both a and b have non-trivial annihilators, i.e., when both a and b are zero-divisors in A_4. This case is explicitly handled in Theorem 7.5.

Finally, Section 8 provides some general results about zero-divisors with very large annihilators. Recall from [DDD] that the largest annihilators in A_n are $(2^n - 4n + 4)$-dimensional.

Definition 1.6. *Let $n \geq 4$, and let c be a multiple of 4 such that $0 \leq c \leq 2^n - 4n$. The space $\boldsymbol{T_n^c}$ is the space of elements of length one in A_n whose annihilators have dimension at least $(2^n - 4n + 4) - c$.*

In other words, T_n^c consists of the zero-divisors with annihilators whose dimensions are within c of the maximum.

Theorem 1.7. *Let $n \geq 4$, and let c be a multiple of 4 such that $0 \leq c \leq 2^n - 4n$. If $n \geq \frac{c}{4} + 4$, then T_{n+1}^c is equal to the space of elements of the form $\{a, 0\}$ or $\{0, a\}$ such that a belongs to T_n^c.*

Theorem 1.7, which is proved in Section 8, tells us that for sufficiently large n, the space T_{n+1}^c is diffeomorphic to the disjoint union of two copies of T_n^c. The case $c = 0$ was proved in [DDD, Theorem 15.7]. An interesting open question is to determine explicitly the geometry of a connected component of T_n^c for n sufficiently large; this connected component depends only on c.

2. Cayley-Dickson algebras

The **Cayley-Dickson algebras** are a sequence of non-associative $\mathbb{R}$-algebras with involution. See [DDD] for a full explanation of their basic properties.

These algebras are defined inductively. We start by defining $\boldsymbol{A_0}$ to be $\mathbb{R}$ with trivial conjugation. Given A_{n-1}, the algebra $\boldsymbol{A_n}$ is defined additively to be $A_{n-1} \times A_{n-1}$. Conjugation in A_n is defined by

$$(a, b)^* = (a^*, -b),$$

and multiplication is defined by

$$(a, b)(c, d) = (ac - d^* b, da + bc^*).$$

One can verify directly from the definitions that A_1 is isomorphic to the complex numbers $\mathbb{C}$; A_2 is isomorphic to the quaternions $\mathbb{H}$; and A_3 is isomorphic to the octonions $\mathbb{O}$.

We implicitly view A_{n-1} as the subalgebra $A_{n-1} \times 0$ of A_n.

2.1. **Complex structure.** The element $\boldsymbol{i_n} = (0, 1)$ of A_n enjoys many special properties. One of the primary themes of our long-term project is to fully exploit these special properties.

Let $\boldsymbol{\mathbb{C}_n}$ be the $\mathbb{R}$-linear span of $1 = (1, 0)$ and i_n. It is a subalgebra of A_n isomorphic to $\mathbb{C}$.

Lemma 2.2 (DDD, Proposition 5.3)**.** *Under left multiplication, A_n is a $\mathbb{C}_n$-vector space. In particular, if α and β belong to $\mathbb{C}_n$ and x belongs to A_n, then $\alpha(\beta x) = (\alpha\beta)x$.*

As a consequence, the expression $\alpha\beta x$ is unambiguous; we will usually simplify notation in this way.

The **real part $\mathbf{Re}(\boldsymbol{x})$** of an element x of A_n is defined to be $\frac{1}{2}(x + x^*)$, while the **imaginary part $\mathbf{Im}(\boldsymbol{x})$** is defined to be $x - \mathrm{Re}(x)$.

The algebra A_n becomes a positive-definite real inner product space when we define $\langle \boldsymbol{a}, \boldsymbol{b} \rangle_{\mathbb{R}} = \mathrm{Re}(ab^*)$ [DDD, Proposition 3.2]. If a and b are imaginary and orthogonal, then ab is imaginary. Hence, $ba = b^*a^* = (ab)^* = -ab$. In other words, orthogonal imaginary elements anti-commute. A simple calculation shows that aa^* and a^*a are both equal to $\langle a, a\rangle_{\mathbb{R}}$ for all a in A_n [DDD, Lemma 3.6].

We will need the following slightly technical result.

Lemma 2.3. *Let x and y be elements of A_n such that y is imaginary. Then x and xy are orthogonal.*

Proof. We wish to show that $\mathrm{Re}(x(xy)^*)$ equals zero. This equals $-\mathrm{Re}((xx^*)y)$ by [DDD, Lemmas 2.6 and 2.8], which is zero because y is imaginary and because xx^* is real. □

The real inner product allows us to define a positive-definite Hermitian inner product on A_n by setting $\langle \boldsymbol{a}, \boldsymbol{b}\rangle_{\mathbb{C}}$ to be the orthogonal projection of ab^* onto the subspace $\mathbb{C}_n$ of A_n [DDD, Proposition 6.3]. We say that two elements a and b are **$\mathbb{C}$-orthogonal** if $\langle a, b\rangle_{\mathbb{C}} = 0$.

We will frequently consider the subspace $\mathbb{C}_{\boldsymbol{n}}^{\perp}$ of A_n; it is the orthogonal complement of $\mathbb{C}_n$ (with respect either to the real or to the Hermitian inner product). Note that $\mathbb{C}_n^{\perp}$ is a $\mathbb{C}_n$-vector space; in other words, if a belongs to $\mathbb{C}_n^{\perp}$ and α belongs to $\mathbb{C}_n$, then αa also belongs to $\mathbb{C}_n^{\perp}$ [DDD, Lemma 3.8].

Lemma 2.4 (DDD, Lemmas 6.4 and 6.5)**.** *If a belongs to $\mathbb{C}_n^{\perp}$, then left multiplication by a is $\mathbb{C}_n$-conjugate-linear in the sense that $a \cdot \alpha x = \alpha^* \cdot ax$ for any x in A_n and any α in $\mathbb{C}_n$. Moreover, left multiplication is anti-Hermitian in the sense that $\langle ax, y\rangle_{\mathbb{C}} = -\langle x, ay\rangle_{\mathbb{C}}^*$.*

Similar results hold for right multiplication by a. See also [M2, Lemma 2.1] for a different version of the claim about conjugate-linearity.

The conjugate-linearity of left and right multiplication is fundamental to many later calculations. To emphasize this point, we provide some computational consequences. The next lemma can be interpreted as a restricted kind of bi-conjugate-linearity for multiplication.

Lemma 2.5. *Let a and b be $\mathbb{C}$-orthogonal elements of $\mathbb{C}_n^{\perp}$, and let α and β belong to $\mathbb{C}_n$. Then $\alpha a \cdot \beta b = \alpha^* \beta^* \cdot ab$.*

Proof. By left conjugate-linearity, $\alpha a \cdot \beta b = \beta^*(\alpha a \cdot b)$. Use right conjugate-linearity twice to compute that $\beta^*(\alpha a \cdot b) = \beta^*(ab \cdot \alpha)$. Because a and b are $\mathbb{C}$-orthogonal, ab belongs to $\mathbb{C}_n^{\perp}$. Therefore, $\beta^*(ab \cdot \alpha) = \beta^*(\alpha^* \cdot ab)$ by left conjugate-linearity again. Finally, this equals $\alpha^*\beta^* \cdot ab$ by Lemma 2.2. □

Norms of elements in A_n are defined with respect to either the real or Hermitian inner product: $|a| = \sqrt{\langle a, a\rangle_{\mathbb{R}}} = \sqrt{\langle a, a\rangle_{\mathbb{C}}} = \sqrt{aa^*}$; this makes sense because aa^* is always a non-negative real number [DDD, Lemma 3.6]. Note also that $|a| = |a^*|$ for all a. We will frequently use that $a^2 = -|a|^2$ if a is an imaginary element of A_n.

Lemma 2.6. *Let a belong to $\mathbb{C}_n^{\perp}$, and let α and β belong to $\mathbb{C}_n$. Then $\alpha a \cdot \beta a = -|a|^2 \alpha\beta^*$.*

Proof. Follow the same general strategy as in the proof of Lemma 2.5. However, instead of using that ab belongs to $\mathbb{C}_n^{\perp}$, use that $a^2 = -|a|^2$ is real. □

One consequence of Lemma 2.6 is that $|\alpha a| = |\alpha||a|$ if α belongs to $\mathbb{C}_n$ and a belongs to $\mathbb{C}_n^{\perp}$. This follows from the computation $\alpha a \cdot \alpha a = -|a|^2 \alpha\alpha^*$.

2.7. **The subalgebra $\mathbb{H}_n$.**

Definition 2.8. *Let* $\mathbb{H}_n$ *be the* $\mathbb{R}$*-linear span of the elements* 1, i_n, i_{n-1}, *and* $i_{n-1}i_n$ *of* A_n.

The notation reminds us that $\mathbb{H}_n$ is a subalgebra isomorphic to the quaternions. Many of the results that follow refer to $\mathbb{H}_n$ and its orthogonal complement $\mathbb{H}_n^\perp$.

In terms of the product $A_n = A_{n-1} \times A_{n-1}$, $\mathbb{H}_n$ is the $\mathbb{R}$-linear span of $(1,0)$, $(0,1)$, $(i_{n-1},0)$, and $(0,i_{n-1})$. By inspection, $\mathbb{H}_n$ is a $\mathbb{C}_n$-linear subspace of A_n. It is also equal to $\mathbb{C}_{n-1} \times \mathbb{C}_{n-1}$. Also, $\mathbb{H}_n^\perp$ and $\mathbb{C}_{n-1}^\perp \times \mathbb{C}_{n-1}^\perp$ are equal as subspaces of A_n.

2.9. **Zero-divisors and annihilators.** A **zero-divisor** is a non-zero element a of A_n such that there exists another non-zero element b in A_n with $ab = 0$. The **annihilator** $\mathbf{Ann}(\boldsymbol{a})$ of a is the set of all elements b such that $ab = 0$. In other words, $\mathrm{Ann}(a)$ is the kernel of left multiplication by a.

Lemma 2.10 (Corollary 1.9 of M1 and Lemma 9.5 of DDD)**.** *If a is a zero-divisor in A_n, then a belongs to $\mathbb{C}_n^\perp$.*

Theorem 2.11 (Theorem 9.8 and Proposition 9.10 of DDD)**.** *The dimension of any annihilator in A_n is a multiple of 4 and is at most $2^n - 4n + 4$.*

See also [M1, Corollary 1.17] for another proof of the first claim.

Lemma 2.12. *Let a belong to $\mathbb{C}_n^\perp$. For any b in A_n, the product ab is orthogonal to* $\mathrm{Ann}(a)$.

Proof. Let c belong to $\mathrm{Ann}(a)$. Use Lemma 2.4 to deduce that $\langle ab, c\rangle_{\mathbb{C}} = -\langle b, ac\rangle_{\mathbb{C}}^*$. This equals zero because $ac = 0$. □

Let $\mathbf{Im}(\boldsymbol{a})$ be the image of left multiplication by a. Lemma 2.12 implies that $\mathrm{Im}(a)$ is the orthogonal complement of $\mathrm{Ann}(a)$ in A_n.

2.13. **Projections.** We still need a few technical definitions and results. We provide complete proofs for the following results because their proofs do not already appear elsewhere.

Definition 2.14. *For any a in A_n, let* $\boldsymbol{\pi_{\mathbb{C}}(a)}$ *be the orthogonal projection of a onto $\mathbb{C}_n$, and let* $\boldsymbol{\pi_{\mathbb{C}}^\perp(a)}$ *be the orthogonal projection of a onto $\mathbb{C}_n^\perp$.*

By definition, $\pi_{\mathbb{C}}(ab^*)$ equals $\langle a, b\rangle_{\mathbb{C}}$ for any a and b.

Lemma 2.15. *Let a and b belong to A_n. Let $b = b' + b''$, where b' is the $\mathbb{C}$-orthogonal projection of b onto the $\mathbb{C}$-linear span of a and where b'' is the $\mathbb{C}$-orthogonal projection of b onto the $\mathbb{C}$-orthogonal complement of a. Then $\pi_{\mathbb{C}}(ab) = ab'$, and $\pi_{\mathbb{C}}^\perp(ab) = ab''$. Similarly, $\pi_{\mathbb{C}}(ba) = b'a$, and $\pi_{\mathbb{C}}^\perp(ba) = b''a$.*

Proof. Note that $ab = ab' + ab''$. The first term belongs to $\mathbb{C}_n$ by Lemma 2.6, while the second term belongs to $\mathbb{C}_n^\perp$ because a and b'' are $\mathbb{C}$-orthogonal. Similarly, $ba = b'a + b''a$, where $b'a$ belongs to $\mathbb{C}_n$ and $b''a$ belongs to $\mathbb{C}_n^\perp$. □

Corollary 2.16. *For any a and b in $\mathbb{C}_n^\perp$, $\pi_{\mathbb{C}}(ab) = \pi_{\mathbb{C}}(ba)^*$ and $\pi_{\mathbb{C}}^\perp(ab) = -\pi_{\mathbb{C}}^\perp(ba)$.*

Proof. Write $b = b' + b''$, where b' is the $\mathbb{C}$-orthogonal projection of b onto a and where b'' is the $\mathbb{C}$-orthogonal projection of b onto the $\mathbb{C}$-orthogonal complement of a. By Lemma 2.15, $\pi_{\mathbb{C}}(ab) = ab'$ and $\pi_{\mathbb{C}}(ba) = b'a$. It follows from Lemma 2.6 that $(ab')^* = b'a$. This finishes the first claim.

For the second claim, Lemma 2.15 implies that $\pi_{\mathbb{C}}^{\perp}(ab) = ab''$ and $\pi_{\mathbb{C}}^{\perp}(ba) = b''a$. Because a and b'' are imaginary and orthogonal, $ab'' = -b''a$. □

Corollary 2.17. *Let a belong to A_n, and let α belong to $\mathbb{C}_n$. Then $\pi_{\mathbb{C}}(\alpha a) = \alpha\pi_{\mathbb{C}}(a) = \pi_{\mathbb{C}}(a\alpha)$.*

Proof. This is an immediate consequence of Lemma 2.15 and the fact that $\mathbb{C}_n$ is commutative. □

One way to interpret Corollary 2.17 is that $\pi_{\mathbb{C}}$ is a $\mathbb{C}$-linear map.

Corollary 2.18. *Let a and b belong to A_n. Then ab belongs to $\mathbb{C}_n$ if and only if b belongs to the $\mathbb{C}$-linear span of a and $\operatorname{Ann}(a)$.*

Proof. In the notation of Lemma 2.15, observe that ab belongs to $\mathbb{C}_n$ if and only if ab'' is zero. □

3. Notation

Definition 3.1. *For any a and b in $\mathbb{C}_n^{\perp}$, let $\boldsymbol{\{a, b\}}$ be the element*

$$\frac{1}{\sqrt{2}}\big(a + b, i_n(-a + b)\big)$$

of A_{n+1}.

Whenever we write an expression of the form $\{a, b\}$, the reader should automatically assume that a and b belong to $\mathbb{C}_n^{\perp}$; nevertheless, we have tried to be explicit with this assumption. The reason for the factors $\frac{1}{\sqrt{2}}$ will show up in Lemma 3.5 and Lemma 4.9, where we study the metric properties of the notation $\{a, b\}$.

Lemma 3.2. *Let (x, y) belong to $\mathbb{H}_{n+1}^{\perp}$, i.e., let x and y belong to $\mathbb{C}_n^{\perp}$. Then*

$$(x, y) = \frac{1}{\sqrt{2}}\big\{x + i_n y, x - i_n y\big\}.$$

The subspace $\mathbb{H}_{n+1}^{\perp}$ of A_{n+1} is equal to the subspace of all elements of the form $\{a, b\}$ with a and b in $\mathbb{C}_n^{\perp}$.

Proof. For the first claim, check the definition. This immediately implies that every element of $\mathbb{H}_{n+1}^{\perp}$ can be written in the form $\{a, b\}$ for some a and b in $\mathbb{C}_n^{\perp}$.

On the other hand, let a and b belong to $\mathbb{C}_n^{\perp}$. Then $a + b$ and $i_n(-a + b)$ also belong to $\mathbb{C}_n^{\perp}$, so $\big(a + b, i_n(-a + b)\big)$ belongs to $\mathbb{H}_{n+1}^{\perp}$. □

Recall that left multiplication makes A_{n+1} into a $\mathbb{C}_{n+1}$-vector space. We now describe multiplication by elements $\mathbb{C}_{n+1}$ with respect to the notation $\{a, b\}$.

Definition 3.3. *If α belongs to $\mathbb{C}_n$, then $\boldsymbol{\tilde{\alpha}}$ is the image of α under the $\mathbb{R}$-linear map $\mathbb{C}_n \to \mathbb{C}_{n+1}$ that takes 1 to 1 and i_n to i_{n+1}.*

Lemma 3.4. *Let a and b belong to $\mathbb{C}_n^{\perp}$, and let α belong to $\mathbb{C}_n$. Then*

$$\tilde{\alpha}\{a, b\} = \{\alpha^* a, \alpha b\}.$$

Proof. Compute directly that $i_{n+1}\{a, 0\} = \{-i_n a, 0\}$ and $i_{n+1}\{0, b\} = \{0, i_n b\}$. □

Lemma 3.5. *For any a and b in $\mathbb{C}_n^\perp$,*

$$|\{a,b\}|^2 = |a|^2 + |b|^2 .$$

Proof. According to Definition 3.1, $|\{a,b\}|^2$ equals

$$\frac{1}{2}\big(|a+b|^2 + |i_n(-a+b)|^2 \big).$$

As a consequence of Lemma 2.6, this expression equals

$$\frac{1}{2}\big(|a+b|^2 + |-a+b|^2 \big),$$

which simplifies to $|a|^2 + |b|^2$ by the parallelogram law. □

The absence of scalars in the above formula is the primary reason that the scalar $\frac{1}{\sqrt{2}}$ appear in Definition 3.1.

4. Multiplication Formulas

This section is the technical heart of the paper. We establish formulas for multiplication with respect to the notation of Section 3. The rest of the paper consists of many applications of these formulas.

Proposition 4.1. *Let a, b, x, and y belong to $\mathbb{C}_n^\perp$, and suppose that a and b are both $\mathbb{C}$-orthogonal to x and y. Then*

$$\{a,b\}\{x,y\} = \sqrt{2}\{ax, by\}.$$

Proof. We begin by computing that $\{a,0\}\{x,0\}$ equals

$$\frac{1}{2}\Big(ax + i_n x \cdot i_n a, -i_n x \cdot a + i_n a \cdot x\Big).$$

Apply Lemma 2.5 to simplify this expression to

$$\frac{1}{2}\Big(ax - xa, i_n \cdot xa - i_n \cdot ax\Big).$$

Note that $ax = -xa$ because a and x are imaginary and orthogonal, so this expression further simplifies to

$$(ax, -i_n \cdot ax),$$

which equals $\sqrt{2}\{ax, 0\}$. A similar calculation shows that

$$\{0,b\}\{0,y\} = \sqrt{2}\{0, by\}.$$

Next we compute that $\{a,0\}\{0,y\}$ equals

$$\frac{1}{2}\Big(ay - i_n y \cdot i_n a, i_n y \cdot a + i_n a \cdot y\Big).$$

Again use Lemma 2.5 to simplify to

$$\frac{1}{2}\Big(ay + ya, -i_n \cdot ya - i_n \cdot ay\Big),$$

but this equals zero because $ay = -ya$.

A similar calculation shows that $\{0,b\}\{x,0\} = 0$. □

Remark 4.2. Proposition 4.1 already gives a sense of how easy it is to express certain zero-divisors using the notation $\{a,b\}$. For example, the product $\{a,0\}\{0,y\}$ is always zero as long as a and y are $\mathbb{C}$-orthogonal elements of $\mathbb{C}_n^\perp$.

Because of the orthogonality hypotheses on a, b, x, and y, Proposition 4.1 does not quite describe how to multiply arbitrary elements of $\mathbb{H}_{n+1}^{\perp}$. Therefore, we need more multiplication formulas to handle various special cases.

Lemma 4.3. *Let a belong to $\mathbb{C}_n^{\perp}$. Then*

$$\{0,a\}\{a,0\} = -\{a,0\}\{0,a\} = |a|^2\,(0,i_n).$$

Proof. Compute that $\{0,a\}\{a,0\}$ equals

$$\tfrac{1}{2}\Big(a^2 - i_na\cdot i_na, -2i_na\cdot a\Big).$$

Lemma 2.6 implies that the first coordinate is zero and that the second coordinate is $|a|^2\, i_n$.

Finally, observe that $\{0,a\}$ and $\{a,0\}$ are orthogonal and imaginary; therefore they anti-commute. □

We write $\tilde{\boldsymbol{\pi}}_{\mathbb{C}}$ for the composition of the projection $A_n \to \mathbb{C}_n$ with the map $\mathbb{C}_n \to \mathbb{C}_{n+1}$ described in Definition 3.3.

Corollary 4.4. *Let a and b be $\mathbb{C}_n$-linearly dependent elements of $\mathbb{C}_n^{\perp}$. Then*

(1) $\{a,0\}\{b,0\} = \tilde{\pi}_{\mathbb{C}}(ab)^*$.
(2) $\{0,a\}\{0,b\} = \tilde{\pi}_{\mathbb{C}}(ab)$.
(3) $\{a,0\}\{0,b\} = \tilde{\pi}_{\mathbb{C}}(ab)\cdot(0,i_n)$.
(4) $\{0,a\}\{b,0\} = -\tilde{\pi}_{\mathbb{C}}(ab)^*\cdot(0,i_n)$.

Proof. Since b belongs to the $\mathbb{C}_n$-linear span of a, we may write $b=\alpha a$ for some α in $\mathbb{C}_n$. Lemma 2.6 implies that ab equals $-|a|^2\,\alpha^*$, so $\tilde{\pi}_{\mathbb{C}}(ab)$ equals $-|a|^2\,\tilde{\alpha}^*$.

On the other hand, $\{a,0\}\{\alpha a,0\}$ equals $\{a,0\}\cdot\tilde{\alpha}^*\{a,0\}$ by Lemma 3.4, which also equals $-|\{a,0\}|^2\,\tilde{\alpha}$ by Lemma 2.6. Finally, this equals $-|a|^2\,\tilde{\alpha}$ by Lemma 3.5. This establishes formula (1). The calculation for formula (2) is similar.

Next, $\{a,0\}\{0,\alpha a\}$ equals $\{a,0\}\cdot\tilde{\alpha}\{0,a\}$ by Lemma 3.4, which also equals $\tilde{\alpha}^*\cdot\{a,0\}\{0,a\}$ by Lemma 2.5. Finally, this equals $-|a|^2\,\tilde{\alpha}^*(0,i_n)$ by Lemma 4.3, establishing formula (3). The calculation for formula (4) is similar. □

We are now ready to give an explicit formula for multiplication of arbitrary elements of $\mathbb{H}_{n+1}^{\perp}$.

Theorem 4.5. *Let a, b, x, and y belong to $\mathbb{C}_n^{\perp}$. Then $\{a,b\}\{x,y\}$ equals*

$$\sqrt{2}\big\{\pi_{\mathbb{C}}^{\perp}(ax),\pi_{\mathbb{C}}^{\perp}(by)\big\} + \tilde{\pi}_{\mathbb{C}}\big(xa+by\big) + \tilde{\pi}_{\mathbb{C}}\big(ay-xb\big)(0,i_n).$$

Proof. We begin by computing $\{a,0\}\{x,0\}$. Write $x = x'+x''$, where x' belongs to the $\mathbb{C}$-linear span of a and x'' is $\mathbb{C}$-orthogonal to a. Then

$$\{a,0\}\{x,0\} = \{a,0\}\{x',0\} + \{a,0\}\{x'',0\}.$$

The first term equals $\tilde{\pi}_{\mathbb{C}}(ax')^*$ by Corollary 4.4, which in turn equals $\tilde{\pi}_{\mathbb{C}}(x'a)$ by Corollary 2.16. This is the same as $\tilde{\pi}_{\mathbb{C}}(xa)$ by Lemma 2.15. The second term equals $\sqrt{2}\{ax'',0\}$ by Proposition 4.1, which equals $\sqrt{2}\{\pi_{\mathbb{C}}^{\perp}(ax),0\}$ by Lemma 2.15. The computation for $\{0,b\}\{0,y\}$ is similar.

Now consider the product $\{a,0\}\{0,y\}$. Write $y = y'+y''$, where y' belongs to the $\mathbb{C}$-linear span of a and y'' is $\mathbb{C}$-orthogonal to a. Then

$$\{a,0\}\{0,y\} = \{a,0\}\{0,y'\} + \{a,0\}\{0,y''\}.$$

The first term equals $\tilde{\pi}_{\mathbb{C}}(ay') \cdot (0, i_n)$ by Corollary 4.4, which is the same as $\tilde{\pi}_{\mathbb{C}}(ay) \cdot (0, i_n)$ by Lemma 2.15. The second term equals zero by Proposition 4.1. The computation for $\{0, b\}\{x, 0\}$ is similar. □

Remark 4.6. The three terms in the formula of Theorem 4.5 are orthogonal. The first term belongs to $\mathbb{H}_{n+1}^{\perp}$; the second term belongs to $\mathbb{C}_{n+1}$; and the third term belongs to $\mathbb{H}_{n+1} \cap \mathbb{C}_{n+1}^{\perp}$, which is also the $\mathbb{C}$-linear span of $(0, i_n)$ or the $\mathbb{R}$-linear span of $(i_n, 0)$ and $(0, i_n)$.

Theorem 4.5 shows how to compute the product of two elements of $\mathbb{H}_{n+1}^{\perp}$. On the other hand, it is easy to multiply elements of $\mathbb{H}_{n+1}$; this is just ordinary quaternionic arithmetic. In order to have a complete description of multiplication on A_{n+1}, we need to explain how to multiply elements of $\mathbb{H}_{n+1}$ with elements of $\mathbb{H}_{n+1}^{\perp}$.

Lemma 3.4 shows how to compute the product of an element of $\mathbb{H}_{n+1}^{\perp}$ and an element of $\mathbb{C}_{n+1}$. It remains only to compute the product of an element of $\mathbb{H}_{n+1}^{\perp}$ and an element of $\mathbb{H}_{n+1} \cap \mathbb{C}_{n+1}^{\perp}$, i.e., the $\mathbb{C}$-linear span of $(0, i_n)$. The following lemma makes this computation.

Lemma 4.7. *Let a and b belong to $\mathbb{C}_n^{\perp}$. Then*

$$(0, i_n)\{a, b\} = -\{a, b\}(0, i_n) = \{b, -a\}.$$

Proof. Compute directly that $(0, i_n)\{a, 0\} = \{0, -a\}$ and that $(0, i_n)\{0, b\} = \{b, 0\}$. Also, use that orthogonal imaginary elements anti-commute. □

4.8. **Inner product computations.**

Lemma 4.9. *Let a, b, x, and y belong to $\mathbb{C}_n^{\perp}$. Then*

$$\big\langle \{a, b\}, \{x, y\} \big\rangle_{\mathbb{C}} = \langle a, x\rangle_{\mathbb{C}}^* + \langle b, y\rangle_{\mathbb{C}}.$$

Proof. We need to compute the projection of the product $-\{a, b\}\{x, y\}$ onto $\mathbb{C}_{n+1}$. Theorem 4.5 immediately shows that this projection equals $-\tilde{\pi}_{\mathbb{C}}(xa + by)$, which is equal to $\langle x, a\rangle_{\mathbb{C}} + \langle b, y\rangle_{\mathbb{C}}$. Finally, recall that $\langle x, a\rangle_{\mathbb{C}} = \langle a, x\rangle_{\mathbb{C}}^*$. □

Corollary 4.10. *Let a, b, x, and y belong to $\mathbb{C}_n^{\perp}$. Then*

$$\big\langle \{a, b\}, \{x, y\} \big\rangle_{\mathbb{R}} = \langle a, x\rangle_{\mathbb{R}} + \langle b, y\rangle_{\mathbb{R}}.$$

Proof. Use Lemma 4.9, recalling that the real inner product equals the real part of the Hermitian inner product. □

4.11. **Subalgebras.** Suppose that a and b are $\mathbb{C}$-orthogonal elements of $\mathbb{C}_n^{\perp}$ that both have norm 1. Suppose also that a and b satisfy the equations $a \cdot ab = -\lambda b$ and $b \cdot ba = -\lambda a$ for some non-zero real number λ. These equations guarantee that a and b generate a 4-dimensional subalgebra of A_n; the subalgebra is isomorphic to $\mathbb{H}$ when $\lambda = 1$. This remark concerns the possible values for λ, and therefore addresses the classification problem for 4-dimensional subalgebras of Cayley-Dickson algebras. See [CD, Section 7] for detailed information on 4-dimensional subalgebras of A_4. In particular, in A_4, the only possible values for λ are 1 and 2 [CD, Theorem 7.1].

Given a and b as in the previous paragraph, compute that

$$\frac{1}{\sqrt{2}}\{a, b\} \cdot \frac{1}{\sqrt{2}}\{b, -a\} = \frac{1}{\sqrt{2}}\{ab, -ba\} + (0, i_n)$$

using Theorem 4.5. Next, compute that

$$\frac{1}{\sqrt{2}}\{a,b\}\left(\frac{1}{\sqrt{2}}\{ab,-ba\}+(0,i_n)\right)=-\frac{\lambda+1}{\sqrt{2}}\{b,-a\}$$

using Proposition 4.1 and Lemma 4.7. This uses that a and ab are $\mathbb{C}$-orthogonal by Lemma 2.3 and also the equations involving a, b, and λ. A similar calculation can be performed with the roles of $\frac{1}{\sqrt{2}}\{a,b\}$ and $\frac{1}{\sqrt{2}}\{b,-a\}$ switched.

We have shown that $\frac{1}{\sqrt{2}}\{a,b\}$ and $\frac{1}{\sqrt{2}}\{b,-a\}$ satisfy the same equations as a and b do, except that λ is replaced by $\lambda+1$. Using the argument of [CD, Theorem 7.1] (which can be applied even when $n>4$), it follows that for every positive integer r and every sufficiently large n (depending on r), there is a subalgebra of A_n that is isomorphic to the non-associative algebra with $\mathbb{R}$-basis $\{1,x,y,z\}$ subject to the multiplication rules

$$x^2=y^2=z^2=-1,\quad xy=-yx=z\sqrt{r},\quad yz=-zy=x\sqrt{r},\quad zx=-xz=y\sqrt{r}.$$

This algebra is isomorphic to $\mathbb{H}$ when $r=1$.

Another consequence of our multiplication formulas is the following observation about sets of mutually annihilating elements.

Lemma 4.12. *Let $n\geq 3$. If $\mathbb{C}_n^\perp$ contains two sets $\{x_1,\ldots,x_{2^{n-3}}\}$ and $\{y_1,\ldots,y_{2^{n-3}}\}$ of size 2^{n-3} such that $x_ix_j=0=y_iy_j$ for all $i\neq j$ and each x_i is $\mathbb{C}$-orthogonal to each y_j, then the product $\{x_i,0\}\{x_j,0\}$ is zero when $i\neq j$, and $\{x_i,0\}\{0,y_j\}$ is zero for all i and j.*

Proof. Compute with Proposition 4.1. □

Corollary 4.13. *The space $\mathbb{C}_n^\perp$ contains 2^{n-3} elements such that the product of any two distinct elements is zero.*

Proof. We will actually prove a stronger result that $\mathbb{C}_n^\perp$ contains two sets $\{x_1,\ldots,x_{2^{n-3}}\}$ and $\{y_1,\ldots,y_{2^{n-3}}\}$ of size 2^{n-3} such that $x_ix_j=0=y_iy_j$ for all $i\neq j$ and each x_i is $\mathbb{C}$-orthogonal to each y_j.

The proof is by induction on n, using Lemma 4.12. The base case $n=3$ is trivial; it just calls for the existence of two orthogonal elements of the six-dimensional subspace $\mathbb{C}_3^\perp$ of A_3.

Now suppose for induction that the sets $\{x_1,\ldots,x_{2^{n-3}}\}$ and $\{y_1,\ldots,y_{2^{n-3}}\}$ exist in A_n. Consider the subset of A_{n+1} consisting of all elements of the form $\{x_i,0\}$ or $\{0,y_j\}$. There are 2^{n-2} such elements, and Lemma 4.12 implies that the product of any two distinct such elements is zero.

Also consider the subset of A_{n+1} consisting of all elements of the form $\{y_j,0\}$ or $\{0,x_i\}$. Again, there are 2^{n-2} such elements, and the product of any two distinct such elements is zero.

Finally, by Lemma 4.9 and the induction assumption, the elements described in the previous two paragraphs are $\mathbb{C}$-orthogonal. □

Corollary 4.13 is also relevant to subalgebras of Cayley-Dickson algebras. The $\mathbb{R}$-linear span of 1 together with a set of mutually annihilating elements is a subalgebra of A_n. These subalgebras are highly degenerate in the sense that $xy=0$ for any pair of orthogonal imaginary elements. Corollary 4.13 implies that A_n contains such a subalgebra of dimension $1+2^{n-3}$. In fact, we have shown that A_n contains two such subalgebras whose imaginary parts are $\mathbb{C}$-orthogonal.

Question 4.14. *Does A_n contain a degenerate subalgebra of dimension larger than $1+2^{n-3}$?*

5. Annihilation in $\mathbb{H}_{n+1}^{\perp}$

In this section, we apply the multiplication formulas of Section 4 to consider zero-divisors in A_{n+1}.

Proposition 5.1. *Let a, b, x, and y belong to $\mathbb{C}_n^{\perp}$. Then $\{a,b\}\{x,y\}=0$ if and only if*

(i) $\pi_{\mathbb{C}}^{\perp}(ax)=0$,
(ii) $\pi_{\mathbb{C}}^{\perp}(by)=0$,
(iii) $xa+by=0$, *and*
(iv) $\pi_{\mathbb{C}}(ay-xb)=0$.

Proof. Parts (i), (ii), and (iv) are immediate from Theorem 4.5. It follows from (i) and (ii) that $\pi_{\mathbb{C}}(xa+by)=xa+by$. Therefore, part (iii) also follows from Theorem 4.5. □

The conditions of Proposition 5.1 are redundant. For example, condition (i) follows from conditions (ii) and (iii). However, it is more convenient to formulate the proposition symmetrically.

Proposition 5.2. *Let $n \geq 3$. Let a and b be non-zero elements of $\mathbb{C}_n^{\perp}$. Then $\mathbb{H}_{n+1}^{\perp} \cap \operatorname{Ann}\{a,b\}$ is equal to the space of all $\{\alpha a+x, \beta b+y\}$ such that:*

1. *x belongs to $\operatorname{Ann}(a)$, and y belongs to $\operatorname{Ann}(b)$;*
2. *α and β belong to $\mathbb{C}_n$;*
3. *$|a|^2\,\alpha + |b|^2\,\beta^* = 0$; and*
4. *$(\beta^*-\alpha)\pi_{\mathbb{C}}(ab)+\pi_{\mathbb{C}}(ay-xb)=0$.*

Proof. We want to solve the equation $\{a,b\}\{z,w\}=\{0,0\}$ under the assumption that z and w belong to $\mathbb{C}_n^{\perp}$ (see Lemma 3.2). Using Proposition 5.1, this is equivalent to solving the four equations

$$\pi_{\mathbb{C}}^{\perp}(az) = 0 \tag{5.3}$$

$$\pi_{\mathbb{C}}^{\perp}(bw) = 0 \tag{5.4}$$

$$za+bw = 0 \tag{5.5}$$

$$\pi_{\mathbb{C}}(aw-zb) = 0. \tag{5.6}$$

By Corollary 2.18, Equations (5.3) and (5.4) are the same as requiring that z belongs to the $\mathbb{C}$-linear span of a and $\operatorname{Ann}(a)$ and that w belongs to the $\mathbb{C}$-linear span of b and $\operatorname{Ann}(b)$. Therefore, we may write $z=\alpha a+x$ and $w=\beta b+y$ for some α and β in $\mathbb{C}_n$, some x in $\operatorname{Ann} a$, and some y in $\operatorname{Ann} b$. We also know that x and y belong to $\mathbb{C}_n^{\perp}$ by Lemma 2.10; this is where we use that a and b are non-zero.

Substitute the expressions for z and w in Equations (5.5) and (5.6) to obtain

$$(\alpha a+x)a+b(\beta b+y) = 0 \tag{5.7}$$

$$\pi_{\mathbb{C}}\Big(a(\beta b+y)-(\alpha a+x)b\Big) = 0. \tag{5.8}$$

Equation (5.7) simplifies to $-\,|a|^2\,\alpha - |b|^2\,\beta^*=0$ by Lemma 2.6 and the fact that $xa=by=0$. This is condition (3) of the proposition.

Equation (5.8) can be rewritten as

$$\pi_{\mathbb{C}}(\beta^* \cdot ab - ab \cdot \alpha) + \pi_{\mathbb{C}}(ay - xb) = 0 \tag{5.9}$$

by Lemma 2.4. Apply Corollary 2.17 to the second part of the first term of Equation (5.9) to obtain the equation $(\beta^* - \alpha)\pi_{\mathbb{C}}(ab) + \pi_{\mathbb{C}}(ay - xb) = 0$. This is condition (4) of the proposition. □

Theorem 5.10. *Let $n \geq 3$, and let a and b be non-zero elements of $\mathbb{C}_n^\perp$. Then* $\dim \operatorname{Ann}\{a, b\}$ *equals* $\dim \operatorname{Ann} a + \dim \operatorname{Ann} b$ *or* $\dim \operatorname{Ann} a + \dim \operatorname{Ann} b + 4$.

Proof. First we will use Proposition 5.2 to analyze $\mathbb{H}_{n+1}^\perp \cap \operatorname{Ann}\{a, b\}$. Let V be the space of elements $\{\alpha a + x, \beta b + y\}$ such that α and β belong to $\mathbb{C}_n$, x belongs to $\operatorname{Ann} a$, and y belongs to $\operatorname{Ann} b$. The dimension of V is equal to $\dim \operatorname{Ann} a + \dim \operatorname{Ann} b + 4$. Recall from Lemma 3.4 that for γ in $\mathbb{C}_n$,

$$\tilde{\gamma}\{\alpha a + x, \beta b + y\} = \{\gamma^* \alpha a + \gamma^* x, \gamma \beta b + \gamma y\}.$$

This shows that V is a $\mathbb{C}_n$-vector space, and Condition (3) of Proposition 5.2 is a non-degenerate conjugate-linear equation in the variables α and β. Hence there is a subspace of V of dimension $\dim \operatorname{Ann} a + \dim \operatorname{Ann} b + 2$ that satisfies condition (3).

Condition (4) of Proposition 5.2 is a conjugate-linear equation in the variables α, β, x, and y, which may or may not be non-degenerate and independent of condition (3). This establishes that

$$\dim \operatorname{Ann} a + \dim \operatorname{Ann} b \leq \dim(\mathbb{H}_{n+1}^\perp \cap \operatorname{Ann}\{a, b\}) \leq \dim \operatorname{Ann} a + \dim \operatorname{Ann} b + 2.$$

Lemma 2.10 implies that $\operatorname{Ann}\{a, b\}$ is contained in $\mathbb{C}_{n+1}^\perp$. Note that $\mathbb{H}_{n+1}^\perp$ is a codimension 2 subspace of $\mathbb{C}_{n+1}^\perp$. Therefore, the codimension of $\mathbb{H}_{n+1}^\perp \cap \operatorname{Ann}\{a, b\}$ in $\operatorname{Ann}\{a, b\}$ is at most 2. This establishes the inequality

$$\dim \operatorname{Ann} a + \dim \operatorname{Ann} b \leq \dim \operatorname{Ann}\{a, b\} \leq \dim \operatorname{Ann} a + \dim \operatorname{Ann} b + 4.$$

The desired result follows from Theorem 2.11, which tells us that the dimension of any annihilator is a multiple of 4. □

Theorem 5.10 gives two options for the dimension of $\operatorname{Ann}\{a, b\}$; Section 6 below contains conditions on a and b that distinguish between these two cases.

One might also be concerned that Theorem 5.10 applies only to elements $\{a, b\}$ in which both a and b are non-zero because it relies on Proposition 5.2. For completeness, we also review from [DDD] the simpler situation of elements of the form $\{a, 0\}$ and $\{0, a\}$. The following proposition can be proved with the formulas of Section 4.

Proposition 5.11 (Theorem 10.2, DDD)**.** *Let $n \geq 4$, and let a belong to $\mathbb{C}_{n-1}^\perp$. Then the element $\{a, 0\}$ of A_n is a zero-divisor whose annihilator* $\operatorname{Ann}\{a, 0\}$ *equals the space of all elements $\{x, y\}$ where x belongs to* $\operatorname{Ann}(a)$ *and y is $\mathbb{C}$-orthogonal to 1 and a. Similarly, the element $\{0, a\}$ of A_n is a zero-divisor whose annihilator* $\operatorname{Ann}\{0, a\}$ *equals the space of all elements $\{x, y\}$ where y belongs to* $\operatorname{Ann}(a)$ *and x is $\mathbb{C}$-orthogonal to 1 and a. In either case, the dimension of the annihilator is* $\dim \operatorname{Ann}(a) + 2^{n-1} - 4$.

In fact, [DDD, Theorem 10.2] was a major inspiration for the notation $\{a, b\}$.

6. The D-locus

In Section 5, we started to consider $\mathrm{Ann}\{a,b\}$ when a and b are arbitrary elements in $\mathbb{C}_n^\perp$, i.e., when $\{a,b\}$ is an arbitrary element of $\mathbb{H}_{n+1}^\perp$. Theorem 5.10 told us that except for some simple well-understood cases covered in Proposition 5.11, the dimension of $\mathrm{Ann}\{a,b\}$ is either $\dim \mathrm{Ann}\, a + \dim \mathrm{Ann}\, b$ or $\dim \mathrm{Ann}\, a + \dim \mathrm{Ann}\, b + 4$. The goal of this section is to distinguish between these two cases.

Definition 6.1. *The **D-locus** is the space of all elements $\{a,b\}$ of A_{n+1} with a and b in $\mathbb{C}_n^\perp$ such that*

(1) a and b are $\mathbb{C}$-orthogonal,
(2) a and $\mathrm{Ann}(b)$ are orthogonal, and
(3) b and $\mathrm{Ann}(a)$ are orthogonal.

Remark 6.2. Since $\mathrm{Ann}(b)$ is a $\mathbb{C}$-subspace of A_n, a is orthogonal to $\mathrm{Ann}(b)$ if and only if a is $\mathbb{C}$-orthogonal to $\mathrm{Ann}(b)$. Similarly, b is orthogonal to $\mathrm{Ann}(a)$ if and only if b is $\mathbb{C}$-orthogonal to $\mathrm{Ann}(a)$. Thus, conditions (2) and (3) of Definition 6.1 can be rewritten in terms of $\mathbb{C}$-orthogonality.

Also, $\mathrm{Ann}(b)^\perp$ is equal to the image of left multiplication by b (see Lemma 6.9 below), so condition (2) is also equivalent to requiring that $a = bx$ for some x. Similarly, condition (3) is also equivalent to requiring that $b = ay$ for some y.

The point of the following lemma is to determine precisely when condition (4) of Proposition 5.2 vanishes.

Lemma 6.3. *Suppose that a and b belong to $\mathbb{C}_n^\perp$. Then $\{a,b\}$ belongs to the D-locus if and only if*

$$(\beta^* - \alpha)\pi_{\mathbb{C}}(ab) + \pi_{\mathbb{C}}(ay - xb) = 0$$

for all α and β in $\mathbb{C}_n$, x in $\mathrm{Ann}(a)$, and y in $\mathrm{Ann}(b)$.

Proof. Since α, β, x, and y are independent, the displayed expression vanishes if and only if $\pi_{\mathbb{C}}(ab) = 0$, $\pi_{\mathbb{C}}(xb) = 0$ for all x in $\mathrm{Ann}\, a$, and $ay = 0$ for all y in $\mathrm{Ann}\, b$. The first equation just means that a and b are $\mathbb{C}$-orthogonal, the second equation means that b is $\mathbb{C}$-orthogonal to $\mathrm{Ann}(a)$, and the third equation means that a is $\mathbb{C}$-orthogonal to $\mathrm{Ann}(b)$. □

Lemma 6.4. *If $\{a,b\}$ is non-zero and does not belong to the D-locus, then the dimension of $\mathrm{Ann}\{a,b\} \cap \mathbb{H}_{n+1}^\perp$ is equal to $\dim \mathrm{Ann}(a) + \dim \mathrm{Ann}(b)$.*

Proof. Let V be the subspace of A_{n+1} consisting of all elements of the form $\{\alpha a + x, \beta b + y\}$, where α and β belong to $\mathbb{C}_n$, x belongs to $\mathrm{Ann}(a)$, and y belongs to $\mathrm{Ann}(b)$. The dimension of V is $\dim \mathrm{Ann}(a) + \dim \mathrm{Ann}(b) + 4$. As in the proof of Theorem 5.10, V is a $\mathbb{C}_n$-vector space.

According to Proposition 5.2, $\mathrm{Ann}\{a,b\} \cap \mathbb{H}_{n+1}^\perp$ is contained in V. In fact, it is the subspace of V defined by the two conjugate-linear equations

$$|\, a\,|^2\, \alpha + |\, b\,|^2\, \beta^* = 0 \tag{6.5}$$

$$(\beta^* - \alpha)\pi_{\mathbb{C}}(ab) + \pi_{\mathbb{C}}(ay - xb) = 0. \tag{6.6}$$

Thus, we only need to show that Equations (6.5) and (6.6) are non-degenerate and independent. Equation (6.5) is non-degenerate because $|\, a\,|$ or $|\, b\,|$ is non-zero. Equation (6.6) is non-degenerate by Lemma 6.3.

It remains to show that Equations (6.5) and (6.6) are independent. There are three cases to consider, depending on which part of Definition 6.1 fails to hold for a and b.

If a and b are not $\mathbb{C}$-orthogonal, then $\pi_{\mathbb{C}}(ab)$ is non-zero. Substitute the values $\alpha = -|b|^2$, $\beta = |a|^2$, $x = 0$, and $y = 0$ into the two equations; note that Equation (6.5) is satisfied, while Equation (6.6) is not satisfied because the left-hand side equals $(|a|^2 + |b|^2)\pi_{\mathbb{C}}(ab)$. This shows that the two equations are independent because they have different solution sets.

Next, suppose that a is not orthogonal to $\mathrm{Ann}(b)$. There exists an element y_0 of $\mathrm{Ann}(b)$ such that a and y_0 are not $\mathbb{C}$-orthogonal. This means that $\pi_{\mathbb{C}}(ay_0)$ is non-zero. Substitute the values $\alpha = 0$, $\beta = 0$, $x = 0$, and $y = y_0$ into the two equations; note that Equation (6.5) is satisfied, while Equation (6.6) is not satisfied because the left-hand side equals $\pi_{\mathbb{C}}(ay_0)$. This shows that the two equations are independent because they have different solution sets.

Finally, suppose that b is not orthogonal to $\mathrm{Ann}(a)$. Similarly to the previous case, choose x_0 in $\mathrm{Ann}(a)$ such that $\pi_{\mathbb{C}}(ax_0)$ is non-zero. Substitute the values $\alpha = 0$, $\beta = 0$, $x = x_0$, and $y = 0$ into the two equations; note that Equation (6.5) is satisfied, while Equation (6.6) is not satisfied. □

Theorem 6.7. *Let a and b be non-zero elements of $\mathbb{C}_n^\perp$. If $\{a, b\}$ does not belong to the D-locus, then $\mathrm{Ann}\{a, b\}$ is contained in $\mathbb{H}_{n+1}^\perp$. Moreover, the dimension of $\mathrm{Ann}\{a, b\}$ is $\dim \mathrm{Ann}\, a + \dim \mathrm{Ann}\, b$.*

Proof. Recall from Lemma 2.10 that $\mathrm{Ann}\{a, b\}$ is a subspace of $\mathbb{C}_{n+1}^\perp$. Also, $\mathbb{H}_{n+1}^\perp$ is a codimension 2 subspace of $\mathbb{C}_{n+1}^\perp$. Therefore, the codimension of $\mathrm{Ann}\{a, b\} \cap \mathbb{H}_{n+1}^\perp$ in $\mathrm{Ann}\{a, b\}$ is at most 2. Together with Lemma 6.4, this implies that the dimension of $\mathrm{Ann}\{a, b\}$ is at least $\dim \mathrm{Ann}\, a + \dim \mathrm{Ann}\, b$ and at most $\dim \mathrm{Ann}\, a + \dim \mathrm{Ann}\, b + 2$. However, the dimension of $\mathrm{Ann}\{a, b\}$ is a multiple of 4 by Theorem 2.11, so it must equal $\dim \mathrm{Ann}\, a + \dim \mathrm{Ann}\, b$. This shows that $\mathrm{Ann}\{a, b\}$ equals $\mathrm{Ann}\{a, b\} \cap \mathbb{H}_{n+1}^\perp$ because their dimensions are equal; in other words, $\mathrm{Ann}\{a, b\}$ is contained in $\mathbb{H}_{n+1}^\perp$. □

Theorem 6.7 computes the dimension of $\mathrm{Ann}\{a, b\}$ for any $\{a, b\}$ that does not belong to the D-locus. However, it leaves something to be desired because it does not explicitly describe $\mathrm{Ann}\{a, b\}$ as a subspace of A_{n+1}. The difficulty arises from our use of the fact that the dimension of $\mathrm{Ann}\{a, b\}$ is a multiple of 4.

Question 6.8. *Describe $\mathrm{Ann}\{a, b\}$ explicitly when $\{a, b\}$ does not belong to the D-locus.*

The rest of this section considers annihilators of elements that belong to the D-locus.

Lemma 6.9. *Suppose that a and b belong to A_n, and suppose that b is orthogonal to $\mathrm{Ann}(a)$. There exists a unique element x such that $ax = b$ and x is orthogonal to $\mathrm{Ann}(b)$.*

Proof. This is a restatement of Lemma 2.12. □

Definition 6.10. *Let a and b belong to A_n, and suppose that b is orthogonal to $\mathrm{Ann}\, a$. Then $\boldsymbol{\frac{b}{a}}$ is the unique element such that $a\frac{b}{a} = b$ and such that $\frac{b}{a}$ is orthogonal to $\mathrm{Ann}\, a$.*

Beware that the definition of $\frac{b}{a}$ is not symmetric. In other words, it is not always true that $\frac{b}{a}a = b$.

Lemma 6.11. *Let a and b be $\mathbb{C}$-orthogonal elements of $\mathbb{C}_n^\perp$, and suppose that b is orthogonal to $\mathrm{Ann}(a)$. Then $\frac{b}{a}$ belongs to $\mathbb{C}_n^\perp$ and is $\mathbb{C}$-orthogonal to both a and b.*

Proof. If $a = 0$, then $\mathrm{Ann}\, a$ is all of A_n so $b = 0$ and $\frac{b}{a}$ also equals 0. In this case, the claim is trivially satisfied. Now assume that a is non-zero.

For the first claim, note that $\langle a, a\frac{b}{a}\rangle_{\mathbb{C}} = \langle a, b\rangle_{\mathbb{C}} = 0$. By Lemma 2.4, this equals $-\langle a^2, \frac{b}{a}\rangle_{\mathbb{C}}^*$. But a^2 is a non-zero real number, so $\frac{b}{a}$ is $\mathbb{C}$-orthogonal to 1 as desired.

Next, note that $a\frac{b}{a} = b$ is orthogonal to $\mathbb{C}_n$, so $\left\langle a, \frac{b}{a}\right\rangle_{\mathbb{C}} = \pi_{\mathbb{C}}(b)$ is zero. Also, compute that

$$\left\langle \tfrac{b}{a}, b\right\rangle_{\mathbb{C}} = \left\langle \tfrac{b}{a}, a\tfrac{b}{a}\right\rangle_{\mathbb{C}} = -\left\langle \left(\tfrac{b}{a}\right)^2, a\right\rangle_{\mathbb{C}}^*$$

using Lemma 2.4. But $\left(\frac{b}{a}\right)^2$ is a real scalar, which is $\mathbb{C}$-orthogonal to a because we assumed that a belongs to $\mathbb{C}_n^\perp$. □

Theorem 6.12. *Let a and b be non-zero elements of $\mathbb{C}_n^\perp$, and suppose that $\{a, b\}$ belongs to the D-locus. Then $\mathrm{Ann}\{a, b\}$ is the $\mathbb{C}$-orthogonal direct sum of:*

1. *the space of all elements $\{x, y\}$ such that x belongs to $\mathrm{Ann}(a)$ and y belongs to $\mathrm{Ann}(b)$;*
2. *the $\mathbb{C}$-linear span of the element $\left\{|b|^2\, a, -\,|a|^2\, b\right\}$;*
3. *the $\mathbb{C}$-linear span of $\left\{\frac{b}{a}, -\frac{a}{b}\right\} + \sqrt{2}(0, i_n)$, where $\frac{b}{a}$ and $\frac{a}{b}$ are described in Definition 6.10.*

In particular, the dimension of $\mathrm{Ann}\{a, b\}$ is equal to $\dim(\mathrm{Ann}\, a) + \dim(\mathrm{Ann}\, b) + 4$.

Proof. It follows from Proposition 5.2 that $\mathrm{Ann}\{a, b\}$ contains the space described in part (1). Recall that Lemma 6.3 implies that condition (4) of Proposition 5.2 vanishes.

Next, note that $\left\{|b|^2\, a, -\,|a|^2\, b\right\}$ satisfies the conditions of Proposition 5.2. It corresponds to $\alpha = |b|^2$, $\beta = -\,|a|^2$, $x = 0$, and $y = 0$.

Finally, we want to show that $\{a, b\}\left\{\frac{b}{a}, -\frac{a}{b}\right\} + \sqrt{2}\{a, b\}(0, i_n)$ is zero. Lemma 6.11 says that Proposition 4.1 applies to the first term, which therefore equals $\sqrt{2}\left\{a\frac{b}{a}, -b\frac{a}{b}\right\}$. This simplifies to $\sqrt{2}\{b, -a\}$. Lemma 4.7 lets us compute that the second term is $\sqrt{2}\{-b, a\}$, as desired.

We have now exhibited a subspace of $\mathrm{Ann}\{a, b\}$ whose dimension is $\dim \mathrm{Ann}\, a + \dim \mathrm{Ann}\, b + 4$. Theorem 5.10 implies that we have described the entire annihilator.

Recall that Lemma 4.9 describes how to compute Hermitian inner products. Using this lemma, parts (1) and (2) are $\mathbb{C}$-orthogonal because a and b are $\mathbb{C}$-orthogonal to $\mathrm{Ann}(a)$ and $\mathrm{Ann}(b)$ respectively. Parts (1) and (3) are $\mathbb{C}$-orthogonal by Definition 6.10. Parts (2) and (3) are $\mathbb{C}$-orthogonal by Lemma 6.11. □

7. The D-locus in A_5

The goal of this section is to explicitly understand the D-locus in A_5 (see Definition 6.1). Unlike most of the rest of this paper, this section uses computational techniques that apply in A_4 but have not yet been made to work in general.

Let us consider whether elements of the form $\{a, 0\}$ belong to the D-locus. If a is non-zero, then part (2) of Definition 6.1 fails. Therefore, $\{a, 0\}$ belongs to the D-locus only if $a = 0$. Similarly, $\{0, b\}$ belongs to the D-locus only if $b = 0$.

From now on, we may suppose that a and b are non-zero. If b is not a zero-divisor, then it is easy to determine whether $\{a, b\}$ belongs to the D-locus. Namely, b must be $\mathbb{C}$-orthogonal to a and to $\mathrm{Ann}(a)$ because condition (2) of Definition 6.1 is vacuous. By symmetry, a similar description applies when a is not a zero-divisor. Since annihilators in A_4 are well-understood [KY, Section 3.2] [M1, Corollary 2.14] [DDD, Sections 11 and 12]), it is relatively straightforward to completely describe the elements $\{a, b\}$ belonging to the D-locus in A_5 such that a or b is not a zero-divisor.

There is only one remaining case to consider. It consists of elements of the form $\{a, b\}$, where a and b are both zero-divisors in A_4. We will focus on such elements in the rest of this section. First we need some preliminary calculations in A_4.

Lemma 7.1. *Let a belong to $\mathbb{C}_3^\perp$. If b is $\mathbb{C}$-orthogonal to 1 and a, and α belongs to $\mathbb{C}_3$, then the element $\{a, 0\}$ of A_4 is orthogonal to the annihilator of $\{b, \alpha a\}$.*

Proof. Let c be the element of A_3 such that $bc = a$; in other words, $c = -\frac{1}{|b|^2} ba$. Note that c is $\mathbb{C}$-orthogonal to both a and b by Lemma 2.3.

Using Proposition 4.1, compute that $\{b, \alpha a\}\{\frac{1}{\sqrt{2}} c, 0\} = \{a, 0\}$. Finally, use Lemma 2.12 to conclude that $\{a, 0\}$ is orthogonal to $\mathrm{Ann}\{b, \alpha a\}$. □

Lemma 7.2. *Let a be a non-zero element of $\mathbb{C}_3^\perp$. If a zero-divisor in A_4 is $\mathbb{C}$-orthogonal to $\{a, 0\}$ and is orthogonal to $\mathrm{Ann}\{a, 0\}$, then it is of the form $\{b, \alpha a\}$, where b is $\mathbb{C}$-orthogonal to a and α belongs to $\mathbb{C}$.*

Proof. Suppose that x is a zero-divisor in A_4 that is $\mathbb{C}$-orthogonal to $\{a, 0\}$ and is orthogonal to $\mathrm{Ann}\{a, 0\}$. Write x in the form $\{b, c\} + (\beta, \gamma)$, where b and c belong to $\mathbb{C}_3^\perp$ while β and γ belong to $\mathbb{C}_3$.

Recall from [DDD, Theorem 10.2] that $\mathrm{Ann}\{a, 0\}$ consists of elements of the form $\{0, y\}$, where y is any element of A_3 that is $\mathbb{C}$-orthogonal to 1 and to a. Since x is orthogonal to $\mathrm{Ann}\{a, 0\}$, Lemma 4.9 implies that c is $\mathbb{C}$-orthogonal to all such y. In other words, c belongs to the $\mathbb{C}$-linear span of a; i.e., $c = \alpha a$ for some α in $\mathbb{C}_3$.

Since $\{a, 0\}$ and x are $\mathbb{C}$-orthogonal, Lemma 4.9 says that b is $\mathbb{C}$-orthogonal to a. Note, in particular, that b and c are $\mathbb{C}$-orthogonal.

Let $x_1 = b + c + \beta$ and $x_2 = -i_3 b + i_3 c + \gamma$ so that $x = (x_1, x_2)$. Recall from [DDD, Proposition 12.1] that since x is a zero-divisor, x_1 and x_2 are imaginary orthogonal elements of A_3 with the same norm.

Multiplication by i_3 preserves norms in A_3. Since x_1 and x_2 have the same norm, it follows that β and γ have the same norm. This uses that b and c are orthogonal, as we have already shown.

Since x_1 and x_2 are orthogonal, it follows that β and γ are orthogonal. This uses that b and c are each orthogonal to both $i_3 b$ and $i_3 c$ since b and c are $\mathbb{C}$-orthogonal.

Next, since x_1 and x_2 are imaginary, it follows that β and γ are $\mathbb{R}$-scalar multiples of i_3. We have shown that β and γ are both orthogonal and parallel and also have the same norm. It follows that β and γ are both zero. □

Proposition 7.3. *Let a, b, and c belong to $\mathbb{C}_3^\perp$, and suppose that a is non-zero. Suppose also that $\{b, c\}$ is a zero-divisor in A_4. The element $\{\{a, 0\}, \{b, c\}\}$ belongs to the D-locus in A_5 if and only if b is $\mathbb{C}$-orthogonal to a and c belongs to the $\mathbb{C}$-linear span of a.*

Proof. First suppose that b is $\mathbb{C}$-orthogonal to a and c belongs to the $\mathbb{C}$-linear span of a. Lemma 4.9 implies that $\{a, 0\}$ and $\{b, c\}$ are $\mathbb{C}$-orthogonal.

By [DDD, Theorem 10.2], the annihilator of $\{a, 0\}$ consists of elements of the form $\{0, y\}$, where y is $\mathbb{C}$-orthogonal to 1 and a. Therefore, Lemma 4.9 implies that $\{b, c\}$ is orthogonal to $\mathrm{Ann}\{a, 0\}$.

Lemma 7.1 implies that $\{a, 0\}$ is orthogonal to $\mathrm{Ann}\{b, c\}$. This finishes one implication.

For the other implication, suppose that $\{\{a, 0\}, \{b, c\}\}$ belongs to the D-locus in A_5. Lemma 7.2 implies that b is $\mathbb{C}$-orthogonal to a and that c belongs to the $\mathbb{C}$-linear span of a. □

Suppose that $a = (a_1, a_2)$ is a zero-divisor in A_4. We recall from [KY, Section 3.2] [M1, Corollary 2.14] [DDD, Sections 11 and 12] some algebraic properties of a. First of all, a_1 and a_2 are imaginary orthogonal elements of A_3 with the same norm. The $\mathbb{R}$-linear span of 1, a_1, a_2, and a_1a_2 is a 4-dimensional subalgebra $\langle\langle a_1, a_2\rangle\rangle$ of A_3 that is isomorphic to the quaternions. The notation indicates that the subalgebra is generated by a_1 and a_2.

The annihilator $\mathrm{Ann}(a)$ is a four-dimensional subspace of A_4 consisting of all elements of the form $(y, -cy)$, where c is the fixed unit vector with the same direction as a_1a_2 and x ranges over the orthogonal complement of $\langle\langle a_1, a_2\rangle\rangle$. The subspace $\langle\langle a_1, a_2\rangle\rangle \times \langle\langle a_1, a_2\rangle\rangle$ is orthogonal to $\mathrm{Ann}(a)$. Let $\mathbf{Eig_2(a)}$ be the orthogonal complement of $\mathrm{Ann}(a)$ and $\langle\langle a_1, a_2\rangle\rangle \times \langle\langle a_1, a_2\rangle\rangle$. This space consists of all elements of the form (y, cy), where c and x are as above. Direct calculation shows that $\mathrm{Eig}_2(a)$ is equal to the space of all elements b of A_4 such that $a(ab) = -2b$. From this perspective, it is the 2-eigenspace of the composition of left multiplication by a and left multiplication by $a^* = -a$.

Corollary 7.4. *Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be zero-divisors in A_4. Then $\{a, b\}$ belongs to the D-locus in A_5 if and only if b belongs to the $\mathbb{R}$-linear span of $(a_1, -a_2)$, (a_2, a_1), and $\mathrm{Eig}_2(a)$.*

Proof. Since a_1 and a_2 are orthogonal and have the same norm, there exists an imaginary element c of unit length such that $a_2 = ca_1$. There exists an automorphism of A_3 that takes c to $-i_3$. Therefore, we may assume that $c = -i_3$. In other words, we may assume that $a = \{a_1, 0\}$.

Then $\mathrm{Eig}_2(a)$ is equal to the space of all elements of the form $\{y, 0\}$, where y is $\mathbb{C}$-orthogonal to 1 and a. Also, $\{0, a\}$ equals $(a_1, -a_2)$, so the $\mathbb{C}_4$-linear span of $\{0, a\}$ is the same as the $\mathbb{R}$-linear span of $(a_1, -a_2)$ and (a_2, a_1).

Finally, apply Proposition 7.3. □

Recall that $V_2(\mathbb{R}^7)$ is the space of orthonormal 2-frames in $\mathbb{R}^7$. In the following theorem, we identify this space with the space of elements (a_1, a_2) of A_4 such that a_1 and a_2 are orthogonal imaginary unit vectors in A_3.

Theorem 7.5. *Consider the space X consisting of all elements $\{a, b\}$ belonging to the D-locus in A_5 such that a and b are zero-divisors with unit length. Let ξ be the 4-plane bundle over $V_2(\mathbb{R}^7)$ whose unit sphere bundle has total space diffeomorphic to the 14-dimensional compact simply connected Lie group G_2 (see* [DDD, Section 7]*). Then X is diffeomorphic to the unit sphere bundle of $\xi \oplus 2$, where $\xi \oplus 2$ is the fiberwise sum of the vector bundle ξ with the trivial 2-dimensional bundle.*

Proof. First, identify $V_2(\mathbb{R}^7)$ with the space of all zero-divisors in A_4 with unit length. Let η be the bundle over $V_2(\mathbb{R}^7)$ whose fiber over a is the space of all ordered pairs (a, b) such that b is a unit length element of $\mathrm{Eig}_2(a)$. The bundle ξ is also a bundle over $V_2(\mathbb{R}^7)$, but the fiber over a is the space of all ordered pairs (a, b) such that b is a unit length element of $\mathrm{Ann}(a)$.

Using the notation in the paragraphs preceding Corollary 7.4, the isomorphism $\mathrm{Eig}_2(a) \to \mathrm{Ann}(a)\colon (y, cy) \mapsto (y, -cy)$ induces an isomorphism from η to ξ.

Next consider the space of all ordered pairs (a, b) such that a is a unit length zero-divisor and b belongs to the $\mathbb{R}$-span of $(a_1, -a_2)$ and (a_2, a_1), where $a = (a_1, a_2)$. The map that takes (a, b) to a is a trivial 2-plane bundle.

Corollary 7.4 shows that X is the unit sphere bundle of $\eta \oplus 2$. □

Remark 7.6. An obvious consequence of Theorem 7.5 is that X is diffeomorphic to the total space of an S^5-bundle over $V_2(\mathbb{R}^7)$. This bundle is the fiberwise double suspension of the usual S^3-bundle over $V_2(\mathbb{R}^7)$ that is used to construct G_2.

8. Stability

Sections 5 and 6 described many properties of annihilators of elements of the form $\{a, b\}$. This section exploits these properties to study large annihilators, i.e., annihilators in A_n whose dimension is at least 2^{n-1}.

We begin with a result that could have been included in [DDD], but its significance was not apparent at the time.

Theorem 8.1. *Let $n \geq 3$, and let a belong to A_n. If the dimension of* $\mathrm{Ann}(a)$ *is at least 2^{n-1}, then a belongs to $\mathbb{H}_n^\perp$.*

Proof. Let $a = (b, c)$. We claim that b and c are both zero-divisors; otherwise, [DDD, Lemma 9.9] would imply that $\mathrm{Ann}(a)$ has dimension at most $2^{n-1} - 1$. Lemma 2.10 implies that b and c belong to $\mathbb{C}_{n-1}^\perp$. □

Theorem 8.1 is important in the following way. When searching for zero-divisors with large annihilators, i.e., with annihilators whose dimension is at least half the dimension of A_n, one need only look in $\mathbb{H}_n^\perp$. Fortunately, Sections 5 and 6 study zero-divisors in $\mathbb{H}_n^\perp$ in great detail.

Next we show by construction that the bound of Theorem 8.1 is sharp in the sense that there exist elements of A_n that do not belong to $\mathbb{H}_n^\perp$ but whose annihilators have dimension $2^{n-1} - 4$. Recall that an element a of A_n is alternative if $a \cdot ax = a^2 x$ for all x. For every n, there exist elements of A_n that are alternative. For example, a straightforward computation shows that if a is an alternative element of A_{n-1}, then $(a, 0)$ is an alternative element of A_n.

Proposition 8.2. *Let a be any non-zero alternative element of $\mathbb{C}_{n-1}^\perp$ such that $|a| = 1$. Then* $\mathrm{Ann}(i_{n-1}, a)$ *is equal to the set of all elements of the form $(x, ai_{n-1} \cdot x)$ such that x is $\mathbb{C}$-orthogonal to 1 and to a. In particular, the dimension of* $\mathrm{Ann}(i_{n-1}, a)$ *is equal to $2^{n-1} - 4$.*

Proof. Let x be $\mathbb{C}$-orthogonal to both 1 and a. Using Lemma 2.4, compute that the product $(i_{n-1}, a)(x, ai_{n-1} \cdot x)$ is always zero. We have exhibited a subspace of $\mathrm{Ann}(i_{n-1}, a)$ that has dimension $2^{n-1} - 4$. By Theorem 8.1, this subspace must be equal to $\mathrm{Ann}(i_{n-1}, a)$. □

A proof of Proposition 8.2 also appears in [M3, Theorem 4.4].

Question 8.3. *Find all of the elements of A_n that have annihilators of dimension $2^{n-1} - 4$.*

The paper [DDD] began an exploration of the largest annihilators in A_n. Recall from Theorem 2.11 that the annihilators in A_n have dimension at most $2^n - 4n + 4$. Moreover, Theorem 15.7 of [DDD] gives a complete description of the elements whose annihilators have dimension equal to this upper bound. The rest of this section provides more results in a similar vein.

Definition 8.4. *Let $n \geq 4$, and let c be a multiple of 4 such that $0 \leq c \leq 2^n - 4n$. The space $\boldsymbol{T_n^c}$ is the space of elements of length one in A_n whose annihilators have dimension at least $(2^n - 4n + 4) - c$.*

This is a change in the definition of T_n^c from that used in [DDD]. The elements of T_n^c are unit length zero-divisors whose annihilators are within c dimensions of the largest possible value. The space T_4^0 is diffeomorphic to the Stiefel manifold $V_2(\mathbb{R}^7)$ of orthonormal 2-frames in $\mathbb{R}^7$ [DDD, Section 12].

We have imposed the condition $n \geq 4$ in order to avoid trivial exceptions to our results involving well-known properties of A_n for $n \leq 3$. Also, we have imposed the condition $c \leq 2^n - 4n$ to ensure that every element of T_n^c is always a zero-divisor.

It follows from Lemma 2.10 that T_n^c is contained in $\mathbb{C}_n^\perp$. Thus, if a and b lie in T_n^c, then it makes sense to talk about $\{a, b\}$. Note that if a is in T_n^c then $\{a, 0\}$ and $\{0, a\}$ lie in T_{n+1}^c. This is because, according to Proposition 5.11, both $\mathrm{Ann}\{a, 0\}$ and $\mathrm{Ann}\{0, a\}$ have dimension equal to $\dim \mathrm{Ann}(a) + 2^n - 4$. Consequently, T_{n+1}^c contains a disjoint union of two copies of T_n^c.

Definition 8.5. *The space T_n^c is **stable** if T_{n+1}^c is diffeomorphic to the space of elements of the form $\{a, 0\}$ or $\{0, a\}$ such that a belongs to T_n^c.*

For $n \geq 4$, the space T_n^0 is stable [DDD, Proposition 15.6]; a vastly simpler proof appears below. In fact, our goal is to completely determine which spaces T_n^c are stable.

Proposition 8.6. *Let $n \geq 4$, and let a belong to A_n. If the dimension of* $\mathrm{Ann}(a)$ *is at least $2^n - 8n + 24$, then a is of the form $\{b, 0\}$ or $\{0, b\}$ with b in $\mathbb{C}_{n-1}^\perp$.*

Proof. Suppose that the dimension of $\mathrm{Ann}(a)$ is at least $2^n - 8n + 24$. Note that $2^{n-1} \leq 2^n - 8n + 24$, so the dimension of $\mathrm{Ann}(a)$ is at least 2^{n-1}. By Theorem 8.1, a belongs to $\mathbb{H}_n^\perp$.

Write $a = \{x, y\}$ for some x and y in $\mathbb{C}_{n-1}^\perp$. Assume for contradiction that both x and y are non-zero. By Theorem 5.10, the dimension of $\mathrm{Ann}(a)$ is at most $\dim \mathrm{Ann}(x) + \dim \mathrm{Ann}(y) + 4$. But the dimensions of $\mathrm{Ann}(x)$ and $\mathrm{Ann}(y)$ are at most $2^{n-1} - 4n + 8$ by Theorem 2.11, so the dimension of $\mathrm{Ann}(a)$ is at most $2^n - 8n + 20$. This is a contradiction, so either x or y is zero. □

Proposition 8.7. *If $n \geq 4$, $c \geq 0$, and $n \geq \frac{c}{4} + 4$, then T_n^c is stable.*

Proof. It follows from the inequalities that $c \leq 2^n - 4n$.

Let a belong to T_{n+1}^c. Note that $2^{n+1} - 4n - c \geq 2^{n+1} - 8n + 16$, so $\mathrm{Ann}(a)$ has dimension at least $2^{n+1} - 8(n+1) + 24$. Proposition 8.6 implies that a is of the form $\{b, 0\}$ or $\{0, b\}$. The result then follows directly from Proposition 5.11. □

Lemma 8.8. *For $n \geq 3$, there exists an element $\{a, b\}$ belonging to the D-locus in A_{n+1} such that a and b are elements of $\mathbb{C}_n^\perp$ whose annihilators have dimension $2^n - 4n + 4$.*

Proof. The proof is by induction on n. The base case is $n = 3$. Since every element of A_3 has a trivial annihilator, this case just requires us to choose two $\mathbb{C}$-orthogonal elements from the 6-dimensional space $\mathbb{C}_3^\perp$.

Now suppose that a' and b' are elements of $\mathbb{C}_n^\perp$ whose annihilators have dimension $2^n - 4n + 4$. Suppose also that $\{a', b'\}$ belongs to the D-locus in A_{n+1}.

Consider the elements $a = \{a', 0\}$ and $b = \{b', 0\}$ of A_{n+1}. By Proposition 5.11 and the induction assumption, a and b have annihilators of dimension $2^{n+1} - 4(n+1) + 4$, as desired.

It remains to show that $\{a, b\}$ belongs to the D-locus in A_{n+2}. By Lemma 4.9 and the induction assumption, a and b are $\mathbb{C}$-orthogonal. Proposition 5.11 describes $\mathrm{Ann}(a)$ and $\mathrm{Ann}(b)$. By inspection of this description, b is $\mathbb{C}$-orthogonal to $\mathrm{Ann}(a)$ because b' is $\mathbb{C}$-orthogonal to $\mathrm{Ann}(a')$ by the induction assumption. Similarly, a is $\mathbb{C}$-orthogonal to $\mathrm{Ann}(b)$. □

Lemma 8.9. *For $n \geq 4$, there exist non-zero elements a and b in $\mathbb{C}_n^\perp$ such that $\mathrm{Ann}\{a, b\}$ has dimension $2^{n+1} - 8n + 12$.*

Proof. By Lemma 8.8, there exist non-zero elements of $\mathbb{C}_n^\perp$ such that $\{a, b\}$ belongs to the D-locus in A_{n+1} and such that $\mathrm{Ann}(a)$ and $\mathrm{Ann}(b)$ both have dimension $2^n - 4n + 4$. Now apply Theorem 6.12 to conclude that $\mathrm{Ann}\{a, b\}$ has dimension $2^{n+1} - 8n + 12$. □

Remark 8.10. Lemma 8.9 shows that the bound of Proposition 8.6 is sharp. Substitute $n-1$ for n in the lemma to construct an element of A_n whose annihilator has dimension $2^n - 8n + 20$.

Proposition 8.11. *Let $n \geq 4$, let $c \leq 2^n - 4n$, and let $n \leq \frac{c}{4} + 3$. Then T_n^c is not stable.*

Proof. Note that $2^{n+1} - 4(n+1) + 4 - c \leq 2^{n+1} - 8n + 12$. Now apply Lemma 8.9 to construct an element $\{a, b\}$ belonging to T_{n+1}^c such that both a and b are non-zero. □

Theorem 8.12. *Let $n \geq 4$, and let c be a multiple of 4 such that $0 \leq c \leq 2^n - 4n$. Then T_n^c is stable if and only if $n \geq \frac{c}{4} + 4$.*

Proof. Combine Propositions 8.7 and 8.11. □

We give two illustrations of the theorem.

Corollary 8.13. *The space of zero-divisors in A_5 whose annihilators are 16-dimensional is diffeomorphic to two disjoint copies of $V_2(\mathbb{R}^7)$.*

Proof. Apply Theorem 8.12 with $n = 5$ and $c = 0$. □

Corollary 8.13 is the same as [DDD, Corollary 14.7]. The proof is vastly more graceful than the one in [DDD]. This demonstrates the power of our computational perspective.

Corollary 8.14. *The space of zero-divisors in A_6 whose annihilators are at least 40-dimensional is diffeomorphic to two disjoint copies of the space of zero-divisors in A_5 whose annihilators are at least 12-dimensional.*

References

[A] J. Adem, *Construction of some normed maps*, Bol. Soc. Mat. Mexicana **20** (1975) 59–75.

[CD] K.-C. Chan and D. Ž. Đoković, *Conjugacy classes of subalgebras of the real sedenions*, Canad. Math. Bull. **49** (2006) 492–507.

[Co] F. Cohen, *On the Whitehead square, Cayley-Dickson algebras, and rational functions*, Papers in honor of José Adem, Bol. Soc. Mat. Mexicana **37** (1992) 55–62.

[DDD] D. K. Biss, D. Dugger, and D. C. Isaksen, *Large annihilators in Cayley-Dickson algebras*, Comm. Algebra, to appear.

[ES] P. Eakin and A. Sathaye, *On automorphisms and derivations of Cayley-Dickson algebras*, J. Algebra **129** (1990) 263–278.

[KY] S. Khalil and P. Yiu, *The Cayley-Dickson algebras, a theorem of A. Hurwitz, and quaternions*, Bull. Soc. Sci. Lett. Łódź Sér. Rech. Déform. **24** (1997), 117–169.

[M1] G. Moreno, *The zero divisors of the Cayley-Dickson algebras over the real numbers*, Bol. Soc. Mat. Mexicana **4** (1998) 13–28.

[M2] G. Moreno, *Constructing zero divisors in the higher dimensional Cayley-Dickson algebras*, ArXiv math.RA/0512517, 2005.

[M3] G. Moreno, *Alternative elements in the Cayley-Dickson algebras*, ArXiv math.RA/0404395, 2004.

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