

CLASSIFICATION SPACES OF MAPS IN MODEL CATEGORIES

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ABSTRACT. We correct a mistake in [DK2] and use this to identify homotopy function complexes in a model category with the nerves of certain categories of zig-zags.

1. INTRODUCTION

Let \mathcal{M} be a model category and let $X, Y \in \mathcal{M}$. Consider the category $\mathcal{M}(X, Y)_{\text{Hom}}$ in which an object is a zig-zag of the form

$$X \xleftarrow{\sim} U \rightarrow V \xleftarrow{\sim} Y,$$

where the indicated maps are weak equivalences. A map from $[X \leftarrow U \rightarrow V \leftarrow Y]$ to $[X \leftarrow U' \rightarrow V' \leftarrow Y]$ consists of weak equivalences $U \rightarrow U'$ and $V \rightarrow V'$ making the evident diagram commute. We'll call $\mathcal{M}(X, Y)_{\text{Hom}}$ the **moduli category** of maps from X to Y . The nerve of this category will be called the moduli space of maps, or the *classification space* of maps. Dwyer-Kan proved that this nerve has the correct homotopy type for a homotopy function complex from X to Y (see [DK1, 6.2,8.4]).

If Y is fibrant (or, dually, if X is cofibrant) one might expect to be able to use a simpler category here. Namely, let $\mathcal{M}(X, Y)_{\text{Hom}-f}$ denote the category whose objects are zig-zags

$$X \xleftarrow{\sim} U \longrightarrow Y$$

and where a map from $[X \xleftarrow{\sim} U \longrightarrow Y] \rightarrow [X \xleftarrow{\sim} U' \longrightarrow Y]$ is a weak equivalence $U \rightarrow U'$ making the diagram commute. Notice that there is an inclusion

$$\mathcal{M}(X, Y)_{\text{Hom}-f} \hookrightarrow \mathcal{M}(X, Y)_{\text{Hom}},$$

as the former is just the full subcategory consisting of objects $[X \leftarrow U \rightarrow V \leftarrow Y]$ in which $Y \rightarrow V$ is the identity map.

Theorem 1.1. *If Y is fibrant then $\mathcal{M}(X, Y)_{\text{Hom}-f} \rightarrow \mathcal{M}(X, Y)_{\text{Hom}}$ induces a weak equivalence on nerves.*

Theorem 1.1 was stated in [BDG, Remark 2.7], and was later needed in the paper [DS]. A proof was not given in [BDG], but it is said there that the theorem follows from the arguments in [DK2, 7.2]. Unfortunately, there turns out to be a mistake in [DK2, 7.2], which we describe in Remark 2.3 below. The purpose of this short note is to correct this mistake and to prove Theorem 1.1.

We remark that the categories $\mathcal{M}(X, Y)_{\text{Hom}-f}$ seem to be of some current interest. In addition to their use in [BDG] and [DS], one also finds them appearing in [J].

1.2. Notation. Throughout the paper we will blur the distinction between a category \mathcal{C} and its nerve $N\mathcal{C}$. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is said to be a weak equivalence if it induces a weak equivalence on nerves. The functor is called a homotopy equivalence if there is a functor $G: \mathcal{D} \rightarrow \mathcal{C}$ together with zig-zags of natural transformations between $F \circ G$ and $Id_{\mathcal{D}}$, and between $G \circ F$ and $Id_{\mathcal{C}}$.

Also, if \mathcal{C} is a category and $X, Y \in \text{Ob}(\mathcal{C})$ then we write $\mathcal{C}(X, Y)$ for the set of maps from X to Y .

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2. MODULI CATEGORIES OF MAPS

Let \mathcal{M} be a model category and let $X, Y \in \mathcal{M}$. Write WFib and WCofib for the categories of trivial fibrations and trivial cofibrations. Following [DK2], let $(\text{WCofib})^{-1}\mathcal{M}(\text{WFib})^{-1}(X, Y)$ denote the full subcategory of $\mathcal{M}(X, Y)_{\text{Hom}}$ whose objects are diagrams

$$X \xleftarrow{\sim} U \longrightarrow V \xleftarrow{\sim} Y.$$

Proposition 2.1. *The inclusion $(\text{WCofib})^{-1}\mathcal{M}(\text{WFib})^{-1}(X, Y) \hookrightarrow \mathcal{M}(X, Y)_{\text{Hom}}$ is a homotopy equivalence.*

Proof. Given an object $[X \leftarrow U \rightarrow V \leftarrow Y]$ in $\mathcal{M}(X, Y)_{\text{Hom}}$, functorially factor $U \rightarrow X$ as $U \xrightarrow{\sim} U' \rightarrow X$. Let V' be the pushout of $U' \xleftarrow{\sim} U \rightarrow V$, and note that $V \rightarrow V'$ is a trivial cofibration. Next, functorially factor the composite $Y \rightarrow V'$ as $Y \xrightarrow{\sim} V'' \rightarrow V'$. Let U'' be the pullback of $U' \rightarrow V' \xleftarrow{\sim} V''$. One has the resulting diagram

$$\begin{array}{ccccc} X & \xleftarrow{\sim} & U & \longrightarrow & V & \xleftarrow{\sim} & Y \\ \parallel & & \downarrow \sim & & \downarrow \sim & & \parallel \\ X & \xleftarrow{\sim} & U' & \longrightarrow & V' & \xleftarrow{\sim} & Y \\ \parallel & & \uparrow \sim & & \uparrow \sim & & \parallel \\ X & \xleftarrow{\sim} & U'' & \longrightarrow & V'' & \xleftarrow{\sim} & Y \end{array}$$

Define a functor $F: \mathcal{M}(X, Y)_{\text{Hom}} \rightarrow (\text{WCofib})^{-1}\mathcal{M}(\text{WFib})^{-1}(X, Y)$ by sending the object $[X \leftarrow U \rightarrow V \leftarrow Y]$ to $[X \leftarrow U'' \rightarrow V'' \leftarrow Y]$. Let j denote the inclusion in the statement of the proposition. The above diagram shows that there are zig-zags of natural weak equivalences between jF and the identity, and between Fj and the identity. \square

Define $\mathcal{M}(\text{WFib})^{-1}(X, Y)$ to be the subcategory of $\mathcal{M}(X, Y)_{\text{Hom}-f}$ whose objects are zig-zags $X \xleftarrow{\sim} U \rightarrow Y$.

Proposition 2.2. *The inclusion $\mathcal{M}(\text{WFib})^{-1}(X, Y) \hookrightarrow \mathcal{M}(X, Y)_{\text{Hom}-f}$ is also a homotopy equivalence, provided Y is fibrant.*

Proof. The proof in this case is a little different than the above. Given a diagram

$$X \xleftarrow{\sim} U \longrightarrow Y$$

functorially factor the induced map $U \rightarrow X \times Y$ as $U \xrightarrow{\sim} U' \rightarrow X \times Y$. Since Y is fibrant, the projection $X \times Y \rightarrow X$ is a fibration; so the composite $U' \rightarrow X \times Y \rightarrow X$ is also a fibration (and hence a trivial fibration, by the two-out-of-three property).

Define $F: \mathcal{M}(X, Y)_{\text{Hom}-f} \rightarrow \mathcal{M}(\text{WFib})^{-1}(X, Y)$ by sending $[X \leftarrow U \rightarrow Y]$ to $[X \leftarrow U' \rightarrow Y]$. It is easy to check that this gives a homotopy inverse for the inclusion. \square

Because of the evident commutative square

$$\begin{array}{ccc} (\text{WCofib})^{-1}\mathcal{M}(\text{WFib})^{-1}(X, Y) & \xrightarrow{\sim} & \mathcal{M}(X, Y)_{\text{Hom}} \\ \uparrow & & \uparrow \\ \mathcal{M}(\text{WFib})^{-1}(X, Y) & \xrightarrow{\sim} & \mathcal{M}(X, Y)_{\text{Hom}-f} \end{array}$$

we now know that the right vertical map is a weak equivalence if and only if the left vertical map is so. For the rest of the paper we will concentrate on the left vertical map.

Define another category $\mathcal{M}(X, Y)_{\text{Hom}-tw}$ in the following way. Its objects are again zig-zags $X \xleftarrow{\sim} U \rightarrow V \xleftarrow{\sim} Y$, but now a map from $[X \xleftarrow{\sim} U \rightarrow V \xleftarrow{\sim} Y]$ to $[X \xleftarrow{\sim} U' \rightarrow V' \xleftarrow{\sim} Y]$ is a commutative diagram of the form

$$\begin{array}{ccccc} X & \xleftarrow{\sim} & U & \longrightarrow & V & \xleftarrow{\sim} & Y \\ \parallel & & \downarrow & & \uparrow & & \parallel \\ X & \xleftarrow{\sim} & U' & \longrightarrow & V' & \xleftarrow{\sim} & Y \end{array}$$

Thus, this category has the same objects as $\mathcal{M}(X, Y)_{\text{Hom}}$ but a different collection of morphisms. We think of it as a ‘twisted’ version of $\mathcal{M}(X, Y)_{\text{Hom}}$.

In the same way, define the category $(\text{WCofib})^{-1}\mathcal{M}(\text{WFib})_{tw}^{-1}$; it is the obvious subcategory of $\mathcal{M}(X, Y)_{\text{Hom}-tw}$. Note that there are inclusions

$$\begin{aligned} \mathcal{M}(X, Y)_{\text{Hom}-f} &\hookrightarrow \mathcal{M}(X, Y)_{\text{Hom}-tw} \quad \text{and} \\ \mathcal{M}(\text{WFib})^{-1}(X, Y) &\hookrightarrow (\text{WCofib})^{-1}\mathcal{M}(\text{WFib})_{tw}^{-1}(X, Y). \end{aligned}$$

Write $(\text{WFib} \downarrow X)$ for the category of trivial fibrations with codomain X . A map in this category is a commutative triangle

$$\begin{array}{ccc} U_1 & \xrightarrow{\sim} & U_2 \\ & \searrow \sim & \swarrow \sim \\ & & X \end{array}$$

The category $(Y \downarrow \text{WCofib})$ of trivial cofibrations under Y is defined analogously.

Remark 2.3. We can now explain the mistake in [DK2] referred to in the introduction. Consider the functor

$$K: (\text{WFib} \downarrow X)^{op} \times (Y \downarrow \text{WCofib}) \rightarrow \text{Set} \hookrightarrow s\text{Set}$$

given by $K(U \xrightarrow{\sim} X, Y \xrightarrow{\sim} V) = \mathcal{M}(U, V)$. It is claimed in [DK2, 7.2iii] that the homotopy colimit of this functor (equivalently, the simplicial replacement) is isomorphic to the nerve of $(\text{WCofib})^{-1}\mathcal{M}(\text{WFib})^{-1}$. This is not correct, however. One

readily checks that the homotopy colimit is the nerve of $(\mathrm{WCofib})^{-1}\mathcal{M}(\mathrm{WFib})_{tw}^{-1}$ instead.

Let $c\mathcal{M}$ and $s\mathcal{M}$ denote the categories of cosimplicial and simplicial objects over \mathcal{M} . Recall that these have Reedy model category structures, as described in [H, Sec. 15.3]. Also, recall that for any $Z \in \mathcal{M}$ one has the associated constant cosimplicial and simplicial objects with value Z ; we will also denote these Z , by abuse.

Let $S: \Delta \rightarrow s\mathrm{Set}$ denote the functor $[n] \mapsto \Delta^n$. If K is any simplicial set, let ΔK denote the overcategory $(S \downarrow K)$ —this is the *category of simplices* of K . An object of ΔK is a pair $([n], s: \Delta^n \rightarrow K)$, and the maps are the obvious ones. We use $\Delta^{op}K$ to denote the opposite of this category.

Note that the nerve of ΔK is homotopy equivalent to K . To see why, regard K as a functor $\Delta^{op} \rightarrow \mathrm{Set} \hookrightarrow s\mathrm{Set}$. The nerve of ΔK is the same as the simplicial replacement of this functor, which is also the same as the homotopy colimit. But by [H, Thm. 19.8.7] the homotopy colimit is weakly equivalent to the realization of this functor, which is K itself.

If $QX_* \xrightarrow{\sim} X$ is a Reedy cofibrant resolution of X in $c\mathcal{M}$, note that there is an evident functor

$$\Delta \mathrm{Map}(QX_*, Y) \rightarrow \mathcal{M}(\mathrm{WFib})^{-1}(X, Y)$$

sending $([n], QX_n \rightarrow Y)$ to $X \xleftarrow{\sim} QX_n \rightarrow Y$. Similarly, if $Y \xrightarrow{\sim} RY_*$ is a Reedy fibrant resolution of Y then there is a functor

$$\Delta^{op} \mathrm{Map}(X, RY_*) \rightarrow (\mathrm{WCofib})^{-1}\mathcal{M}(X, Y).$$

The arguments of [DK2] show the following (for the notions of homotopy cofinal, see [H, Def. 19.6.1]):

Theorem 2.4. *Let $QX_* \xrightarrow{\sim} X$ be a Reedy cofibrant resolution of X in $c\mathcal{M}$, and let $Y \xrightarrow{\sim} RY_*$ be a Reedy fibrant resolution of Y in $s\mathcal{M}$.*

- (a) *The functor $Q: \Delta \rightarrow (\mathrm{WFib} \downarrow X)$ is homotopy left cofinal.*
- (b) *The functor $R: \Delta^{op} \rightarrow (Y \downarrow \mathrm{WCofib})$ is homotopy right cofinal.*
- (c) *The map $\Delta \mathrm{Map}(QX_*, Y) \rightarrow \mathcal{M}(\mathrm{WFib})^{-1}(X, Y)$ is a weak equivalence.*
- (d) *The map $\Delta^{op} \mathrm{Map}(X, RY_*) \rightarrow (\mathrm{WCofib})^{-1}\mathcal{M}(X, Y)$ is a weak equivalence.*
- (e) *The map $\Delta \mathrm{diag} \mathrm{Map}(QX_*, RY_*) \rightarrow (\mathrm{WCofib})^{-1}\mathcal{M}(\mathrm{WFib})_{tw}^{-1}(X, Y)$ is a weak equivalence.*

Proof. Although this is mostly in [DK2], we give a brief sketch for the reader's convenience.

For part (a), fix a trivial fibration $U \xrightarrow{\sim} X$. The overcategory $(Q \downarrow [U \rightarrow X])$ has objects pairs $([n], QX_n \rightarrow U)$ where the composite $QX_n \rightarrow U \rightarrow X$ is the same as the fixed map $QX_n \rightarrow X$. This is precisely the category of simplices $\Delta \mathrm{Map}_X(QX_*, U)$ where the maps are being computed in the overcategory $(\mathcal{M} \downarrow X)$. But by [H, Prop. 16.4.6(1)], the map $\mathrm{Map}_X(QX_*, U) \rightarrow \mathrm{Map}_X(QX_*, X)$ is a trivial fibration. Since the codomain is just a point, we have $\mathrm{Map}_X(QX_*, U)$ is contractible. But the nerve of $\Delta \mathrm{Map}_X(QX_*, U)$ is homotopy equivalent to $\mathrm{Map}_X(QX_*, U)$, by the remarks preceding the theorem. Putting everything together, we have shown that $(Q \downarrow [U \rightarrow X])$ has contractible nerve. The proof for (b) is similar.

For (c), consider the functor $F: (\mathrm{WFib} \downarrow X)^{op} \rightarrow \mathrm{Set} \hookrightarrow s\mathrm{Set}$ sending $U \rightarrow X$ to $\mathcal{M}(U, Y)$. The homotopy colimit (or simplicial replacement) of this functor is

precisely the nerve of the category $\mathcal{M}(\mathrm{WFib})^{-1}(X, Y)$. Likewise, the homotopy colimit of the composite

$$\Delta^{op} \xrightarrow{Q^{op}} (\mathrm{WFib} \downarrow X)^{op} \xrightarrow{F} \mathrm{Set} \hookrightarrow s\mathrm{Set}$$

is the nerve of $\Delta \mathrm{Map}(QX_*, Y)$. Using that the functor Q^{op} is homotopy right cofinal, the induced map of homotopy colimits is a weak equivalence by [H, Thm. 19.6.7]. This proves (c), and (d) is similar.

For (e), the functor in the statement is the obvious one which sends a pair $([n], QX_n \rightarrow RY_n)$ to the zig-zag $[X \leftarrow QX_n \rightarrow RY_n \leftarrow Y]$. To prove that this is a weak equivalence, one introduces $F: (\mathrm{WFib} \downarrow X)^{op} \times (Y \downarrow \mathrm{WCofib}) \rightarrow \mathrm{Set}$ sending the pair $[U \rightarrow X], [Y \rightarrow V]$ to $\mathcal{M}(U, V)$. The homotopy colimit of F is the nerve of $(\mathrm{WCofib})^{-1}\mathcal{M}(\mathrm{WFib})_{tw}^{-1}(X, Y)$. Now consider the composite

$$\Delta^{op} \xrightarrow{\mathrm{diag}} \Delta^{op} \times \Delta^{op} \xrightarrow{Q^{op} \times R} (\mathrm{WFib} \downarrow X)^{op} \times (Y \downarrow \mathrm{WCofib}) \xrightarrow{F} \mathrm{Set}.$$

The homotopy colimit of the composite is the nerve of $\Delta(\mathrm{diag} \mathrm{Map}(QX_*, RY_*))$. The result now follows from the fact that the functors diag and $Q^{op} \times R$ are homotopy right cofinal. \square

Corollary 2.5. $\mathcal{M}(\mathrm{WFib})^{-1}(X, Y) \hookrightarrow (\mathrm{WCofib})^{-1}\mathcal{M}(\mathrm{WFib})_{tw}^{-1}(X, Y)$ is a weak equivalence, provided that Y is fibrant.

Proof. Consider the square

$$\begin{array}{ccc} \Delta \mathrm{Map}(QX_*, Y) & \longrightarrow & \mathcal{M}(\mathrm{WFib})^{-1}(X, Y) \\ \downarrow & & \downarrow \\ \Delta \mathrm{diag} \mathrm{Map}(QX_*, RY_*) & \longrightarrow & (\mathrm{WCofib})^{-1}\mathcal{M}(\mathrm{WFib})_{tw}^{-1}(X, Y). \end{array}$$

This square does not commute, but there is a natural transformation from one of the composites to the other—so the induced diagram of nerves commutes in the homotopy category of simplicial sets.

The horizontal maps are weak equivalences by the above proposition. The left vertical map is a weak equivalence by [H, Prop. 17.4.6], using that Y is fibrant. So the right vertical map is also a weak equivalence. \square

This section has been concerned with defining terminology and identifying exactly what is proven in [DK2]. The heart of the paper lies in the next two sections, where our goal is to replace $(\mathrm{WCofib})^{-1}\mathcal{M}(\mathrm{WFib})_{tw}^{-1}(X, Y)$ by $(\mathrm{WCofib})^{-1}\mathcal{M}(\mathrm{WFib})^{-1}(X, Y)$ in the above result.

3. DOUBLE CATEGORIES

The notion of ‘double category’ was introduced by Ehresmann [E]. For a topologist the name ‘bicategory’ is much more appealing, because their relation to categories is analogous to that of bisimplicial sets to simplicial sets. In [W] they are actually *called* bicategories, but unfortunately this term has a rather different meaning among category theorists. Somewhat reluctantly, we will stick to Ehresmann’s original term.

Definition 3.1. A *double category* \mathcal{C} consists of

- (1) A category \mathcal{C}_h whose maps we denote \xrightarrow{h} ,

- (2) A category \mathcal{C}_v with the same object set as \mathcal{C}_h , and whose maps we denote \xrightarrow{v} ,
(3) A collection S of squares of the form

$$\begin{array}{ccc} \bullet & \xrightarrow{h} & \bullet \\ \downarrow v & & \downarrow v \\ \bullet & \xrightarrow{h} & \bullet \end{array}$$

which we call ‘bi-commutative squares’. This collection must contain all squares of the forms

$$\begin{array}{ccc} X & \xrightarrow{Id} & X \\ \alpha \downarrow v & & \alpha \downarrow v \\ X & \xrightarrow{Id} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{h} & X \\ Id \downarrow & & Id \downarrow \\ X & \xrightarrow{\beta} & X \end{array}$$

It must have the property that given two overlapping squares as in

$$\begin{array}{ccccc} \bullet & \xrightarrow{h} & \bullet & \xrightarrow{h} & \bullet \\ \downarrow v & & \downarrow v & & \downarrow v \\ \bullet & \xrightarrow{h} & \bullet & \xrightarrow{h} & \bullet \end{array}$$

if the two smaller squares are in S then so is the outer square. Finally, it must have the analogous property in which the roles of h and v are switched.

Remark 3.2. The information in a category naturally fits into the first two levels of a simplicial set, via the nerve construction. Likewise, the information in a double category naturally fits into the first two levels of a bisimplicial set. See Section 3.3.

Let I be a small double category. We define a functor $F: I \rightarrow \mathcal{C}$ to consist of two ordinary functors $F_h: I_h \rightarrow \mathcal{C}_h$ and $F_v: I_v \rightarrow \mathcal{C}_v$ which have the same behavior on objects and send bicommutative squares to bicommutative squares. The collection $\text{Map}(I, \mathcal{C})$ of such functors itself forms a double category, in the following way. One defines a morphism in $\text{Map}(I, \mathcal{C})_h$ from F_1 to F_2 to be a collection of h -maps $F_1(X) \xrightarrow{h} F_2(X)$ such that for any h -map $X \rightarrow Y$ one gets a commutative square and for any v -map $X \rightarrow Y$ one gets a bicommutative square. Morphisms in $\text{Map}(I, \mathcal{C})_v$ are defined similarly, and the notion of bicommutative square is inherited in the obvious way from \mathcal{C} .

3.3. Nerves of double categories. Let Δ_v^n be the double category whose underlying vertical category is

$$0 \xrightarrow{v} 1 \xrightarrow{v} \cdots \xrightarrow{v} n$$

and in which all h -maps are the identities. Define Δ_h^n similarly.

Let \mathcal{C} be a double category. A **v -chain of length n** is a functor $\Delta_v^n \rightarrow \mathcal{C}$; that is to say, it is a sequence of maps of the form

$$X_0 \xrightarrow{v} X_1 \xrightarrow{v} \cdots \xrightarrow{v} X_n.$$

Write $\text{Map}(\Delta_v^n, \mathcal{C})$ for the double category of v -chains of length n .

Define the **nerve** of \mathcal{C} be the bisimplicial set $N\mathcal{C}_{*,*}$ in which the simplicial set forming the ‘column’ $N\mathcal{C}_{*,q}$ is the usual nerve of the category $\text{Map}(\Delta_v^q, \mathcal{C})_h$. It

follows that the elements of $N\mathcal{C}_{p,q}$ are arrays of bicommutative squares

$$\begin{array}{ccccccc}
 \bullet & \xrightarrow{h} & \bullet & \xrightarrow{h} & \cdots & \xrightarrow{h} & \bullet \\
 \downarrow v & & \downarrow v & & & & \downarrow v \\
 \bullet & \xrightarrow{h} & \bullet & \xrightarrow{h} & \cdots & \xrightarrow{h} & \bullet \\
 \downarrow v & & \downarrow v & & & & \downarrow v \\
 \vdots & & \vdots & & \vdots & & \vdots \\
 \downarrow v & & \downarrow v & & & & \downarrow v \\
 \bullet & \xrightarrow{h} & \bullet & \xrightarrow{h} & \cdots & \xrightarrow{h} & \bullet
 \end{array}$$

in which there are p h -arrows in each row and q v -arrows in each column.

Observe that the row $N\mathcal{C}_{p,*}$ is the usual nerve of the category $\text{Map}(\Delta_h^p, \mathcal{C})_v$. Also, observe that the 0th row of $N\mathcal{C}_{*,*}$ is just $N\mathcal{C}_v$, and the 0th column is $N\mathcal{C}_h$. One obtains two natural maps

$$(3.4) \quad N\mathcal{C}_h \rightarrow \text{diag}(N\mathcal{C}) \quad \text{and} \quad N\mathcal{C}_v \rightarrow \text{diag}(N\mathcal{C}).$$

3.5. Trivial double categories. Suppose \mathcal{C} is an ordinary category. One can define a double category \mathcal{C}_{bi} by setting $\mathcal{C}_h = \mathcal{C}_v = \mathcal{C}$ and letting the bicommutative squares be the ordinary commutative squares. Note that in this situation the two maps of (3.4) are both of the form $N\mathcal{C} \rightarrow \text{diag}(N\mathcal{C}_{bi})$. These are certainly not equal, but the following is true:

Proposition 3.6. *In the above situation, the two maps $N\mathcal{C} \rightarrow \text{diag}(N\mathcal{C}_{bi})$ represent the same map in the homotopy category of simplicial sets.*

Proof. Let f_1 and f_2 be the two maps in the statement of the proposition. We first claim that both f_1 and f_2 are weak equivalences. To see this, note that the n th column (or the n th row) of $N\mathcal{C}_{bi}$ is precisely the nerve of the ordinary diagram category $\text{Map}(\Delta^n, \mathcal{C})$. However, this category is easily seen to be homotopy equivalent to \mathcal{C} itself. So every horizontal (or vertical) map in the bisimplicial set $N\mathcal{C}_{bi}$ is a weak equivalence, and it follows from this that f_1 and f_2 are weak equivalences.

We next claim that there is a map $\chi: \text{diag}(N\mathcal{C}_{bi}) \rightarrow N\mathcal{C}$ which is a splitting for both f_1 and f_2 . To see this, note that an n -simplex in $\text{diag}(N\mathcal{C}_{bi})$ is a commutative diagram

$$(3.7) \quad \begin{array}{ccccccc}
 X_{00} & \longrightarrow & X_{01} & \longrightarrow & \cdots & \longrightarrow & X_{0n} \\
 \downarrow & & \downarrow & & & & \downarrow \\
 X_{10} & \longrightarrow & X_{11} & \longrightarrow & \cdots & \longrightarrow & X_{1n} \\
 \downarrow & & \downarrow & & & & \downarrow \\
 \vdots & \longrightarrow & \vdots & \longrightarrow & \vdots & \longrightarrow & \vdots \\
 \downarrow & & \downarrow & & & & \downarrow \\
 X_{n0} & \longrightarrow & X_{n1} & \longrightarrow & \cdots & \longrightarrow & X_{nn}
 \end{array}$$

(and the face operator d_i just deletes the i th row and column simultaneously, etc.) Our map χ will send this n -simplex to the n -simplex of $N\mathcal{C}$ represented by

$$X_{00} \rightarrow X_{11} \rightarrow X_{22} \rightarrow \cdots \rightarrow X_{nn}.$$

One readily checks that this is a map of simplicial sets. To see that it splits f_1 and f_2 , just observe that f_1 sends a chain

$$Y_0 \rightarrow Y_1 \rightarrow \cdots \rightarrow Y_n$$

to the array as in (3.7) having this chain along each column and all horizontal maps the identities. The map f_2 is similar, but gives an array in which all vertical maps are the identities.

It follows that χ is a weak equivalence, and the fact that $\chi f_1 = \chi f_2$ now shows that f_1 and f_2 represent the same map in the homotopy category. \square

3.8. Reduction of double categories. If \mathcal{C} is a double category then one has natural maps of simplicial sets $N\mathcal{C}_h \rightarrow \text{diag } N\mathcal{C}$ and $N\mathcal{C}_v \rightarrow \text{diag } N\mathcal{C}$. We are interested in conditions forcing these maps to be weak equivalences.

For the following proposition, let Z denote the double category consisting of six objects and a single zig-zag of bicommutative squares as indicated by

$$\begin{array}{ccccc} 0 & \xrightarrow{h} & 1 & \xleftarrow{h} & 2 \\ \downarrow v & & \downarrow v & & \downarrow v \\ 3 & \xrightarrow{h} & 4 & \xleftarrow{h} & 5. \end{array}$$

Proposition 3.9. *Let \mathcal{C} be a double category. Assume that for each v -map $\alpha: X \xrightarrow{v} Y$ there exists an h -functorial zig-zag of bicommutative squares of the form*

$$\begin{array}{ccccc} X & \xrightarrow{h} & \tilde{X} & \xleftarrow{h} & Y \\ \alpha \downarrow v & & \downarrow v & & Id \downarrow v \\ Y & \xrightarrow[h]{Id} & Y & \xleftarrow[h]{Id} & Y. \end{array}$$

Here h -functorial means that the construction is a functor $\text{Map}(\Delta_v^1, \mathcal{C})_h \rightarrow \text{Map}(Z, \mathcal{C})_h$. Then the evident map $N\mathcal{C}_h \rightarrow \text{diag}(N\mathcal{C})$ is a weak equivalence of simplicial sets.

Proof. Consider the functor $F: \text{Map}(\Delta_v^{n-1}, \mathcal{C})_h \rightarrow \text{Map}(\Delta_v^n, \mathcal{C})_h$ sending a sequence

$$X_0 \xrightarrow{v} X_1 \xrightarrow{v} \cdots \xrightarrow{v} X_{n-1}$$

to

$$X_0 \xrightarrow{id} X_0 \xrightarrow{v} X_1 \xrightarrow{v} \cdots \xrightarrow{v} X_{n-1}.$$

Also consider the forgetful functor $U: \text{Map}(\Delta_v^n, \mathcal{C})_h \rightarrow \text{Map}(\Delta_v^{n-1}, \mathcal{C})_h$ sending

$$X_0 \xrightarrow{v} X_1 \xrightarrow{v} \cdots \xrightarrow{v} X_n$$

to

$$X_1 \xrightarrow{v} X_2 \xrightarrow{v} \cdots \xrightarrow{v} X_n.$$

Then $U \circ F$ is the identity, and we claim there is a zig-zag of natural transformations between $F \circ U$ and the identity. This follows immediately from the hypothesis of the proposition. So U and F are homotopy equivalences.

It now follows easily that every horizontal face and degeneracy in $NC_{*,*}$ is a weak equivalence. Thus, $NC_h = NC_{*,0} \rightarrow \text{diag}(NC)$ is a weak equivalence as well. \square

4. APPLICATION TO MODULI CATEGORIES

Let \mathcal{M} be a model category and let $X, Y \in \mathcal{M}$. Let \mathcal{C} be the double category for which

$$\mathcal{C}_h = (\text{WCofib})^{-1}\mathcal{M}(\text{WFib})^{-1}(X, Y) \quad \text{and} \quad \mathcal{C}_v = (\text{WCofib})^{-1}\mathcal{M}(\text{WFib})_{tw}^{-1}(X, Y).$$

Define the bicommutative squares to be all squares which give commutative diagrams in \mathcal{M} . That is, a square involving objects $[X \leftarrow U_i \rightarrow V_i \leftarrow Y]$, $1 \leq i \leq 4$, is called bicommutative if it gives a commutative square when restricted to the ‘ U ’ factors, and also gives a commutative square when restricted to the ‘ V ’ factors.

Lemma 4.1. *For each map $\alpha: A \xrightarrow{v} B$ in \mathcal{C}_v there exists an h -functorial zig-zag of bicommutative squares of the form*

$$\begin{array}{ccccc} A & \xrightarrow{h} & \tilde{A} & \xleftarrow{h} & B \\ \alpha \downarrow v & & \downarrow v & & \text{Id} \downarrow v \\ B & \xrightarrow{h} & B & \xleftarrow{h} & B. \end{array}$$

Moreover, for each map $\beta: X \xrightarrow{h} Y$ in \mathcal{C}_h there exists a v -functorial zig-zag of bicommutative squares of the form

$$\begin{array}{ccccc} X & \xleftarrow{v} & \tilde{X} & \xrightarrow{v} & Y \\ \beta \downarrow h & & \downarrow h & & \text{Id} \downarrow h \\ Y & \xleftarrow{v} & Y & \xrightarrow{v} & Y. \end{array}$$

Proof. Let $A = [X \leftarrow U \rightarrow V \leftarrow Y]$ and $B = [X \leftarrow U' \rightarrow V' \leftarrow Y]$. Let the components of $\alpha: A \xrightarrow{v} B$ be $f: U \rightarrow U'$ and $g: V' \rightarrow V$. Our zig-zag is the evident one of the form

$$\begin{array}{ccccc} [X \leftarrow U \rightarrow V \leftarrow Y] & \xrightarrow{h} & [X \leftarrow U' \rightarrow V \leftarrow Y] & \xleftarrow{h} & [X \leftarrow U' \rightarrow V' \leftarrow Y] \\ \downarrow v & & \downarrow v & & \text{Id} \downarrow v \\ [X \leftarrow U' \rightarrow V' \leftarrow Y] & \xrightarrow{h} & [X \leftarrow U' \rightarrow V' \leftarrow Y] & \xleftarrow{h} & [X \leftarrow U' \rightarrow V' \leftarrow Y]. \end{array}$$

For instance, for the middle object in the top row the map $U' \rightarrow V$ is the composite $U' \rightarrow V' \rightarrow V$. The middle vertical map is the one with components $\text{Id}: U' \rightarrow U'$ and $g: V' \rightarrow V$, the second map in the top row has the same two components, etc.

Similarly, let the components of $\beta: A \xrightarrow{h} B$ be $p: U \rightarrow U'$ and $q: V \rightarrow V'$. Our second zig-zag is the evident one of the form

$$\begin{array}{ccccc} [X \leftarrow U \rightarrow V \leftarrow Y] & \xleftarrow{v} & [X \leftarrow U \rightarrow V' \leftarrow Y] & \xrightarrow{v} & [X \leftarrow U' \rightarrow V' \leftarrow Y] \\ \downarrow h & & \downarrow h & & \text{Id} \downarrow h \\ [X \leftarrow U' \rightarrow V' \leftarrow Y] & \xleftarrow{v} & [X \leftarrow U' \rightarrow V' \leftarrow Y] & \xrightarrow{v} & [X \leftarrow U' \rightarrow V' \leftarrow Y]. \end{array}$$

\square

Corollary 4.2. *The two maps $N\mathcal{C}_h \rightarrow \text{diag}(N\mathcal{C}_{*,*}) \leftarrow N\mathcal{C}_v$ are weak equivalences.*

Proof. The fact that the first map is a weak equivalence follows from the above lemma and Proposition 3.9. For the second map, we use the obvious analog of Proposition 3.9 in which the roles of h and v have been interchanged (and where certain zig-zags have been replaced by zag-zigs). \square

Proposition 4.3. *The map $\mathcal{M}(\text{WFib})^{-1}(X, Y) \hookrightarrow (\text{WCofib})^{-1}\mathcal{M}(\text{WFib})^{-1}(X, Y)$ is a weak equivalence if Y is fibrant.*

Proof. Recall that, in addition to the inclusion j_1 from the statement, one also has the inclusion

$$j_2: \mathcal{M}(\text{WFib})^{-1}(X, Y) \hookrightarrow (\text{WCofib})^{-1}\mathcal{M}(\text{WFib})_{tw}^{-1}(X, Y).$$

One gets a resulting (non-commutative) diagram of simplicial sets

$$\begin{array}{ccc} & N[(\text{WCofib})^{-1}\mathcal{M}(\text{WFib})^{-1}(X, Y)] & \\ & \nearrow^{j_1} & \searrow^{\sim} \\ N[\mathcal{M}(\text{WFib})^{-1}(X, Y)] & & \text{diag}(N\mathcal{C}) \\ & \searrow_{j_2} & \nearrow_{\sim} \\ & N[(\text{WCofib})^{-1}\mathcal{M}(\text{WFib})_{tw}^{-1}(X, Y)] & \end{array}$$

in which the indicated maps are weak equivalences by Corollary 4.2. Moreover, one checks that if we make $\mathcal{M}(\text{WFib})^{-1}(X, Y)$ into a double category in the trivial way (as in Section 3.5) then we actually have a map of double categories

$$F: [\mathcal{M}(\text{WFib})^{-1}(X, Y)]_{bi} \rightarrow \mathcal{C}.$$

The two composites in the above diagram are the same as the two composites

$$N[\mathcal{M}(\text{WFib})^{-1}(X, Y)] \begin{array}{c} \xrightarrow{i_1} \\ \xrightarrow{i_2} \end{array} N[\mathcal{M}(\text{WFib})^{-1}(X, Y)]_{bi} \xrightarrow{F} N\mathcal{C},$$

where i_1 and i_2 are the two maps from (3.4). By Proposition 3.6, i_1 and i_2 represent the same map in the homotopy category of simplicial sets, so the same is true of the two composites in our diagram. Thus, we find that j_1 is a weak equivalence if and only if j_2 is a weak equivalence. But Corollary 2.5 verified that the latter is a weak equivalence. \square

Finally, we complete the proof of our main result:

Proof of Theorem 1.1. This follows immediately from the above result and the diagram immediately after Proposition 2.2. \square

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