EIGENTHEORY OF CAYLEY-DICKSON ALGEBRAS

DANIEL K. BISS, J. DANIEL CHRISTENSEN, DANIEL DUGGER, AND DANIEL C. ISAKSEN

ABSTRACT. We show how eigentheory clarifies many algebraic properties of Cayley-Dickson algebras. These notes are intended as background material for those who are studying this eigentheory more closely.

1. INTRODUCTION

Cayley-Dickson algebras are non-associative finite-dimensional $\mathbb{R}$-division algebras that generalize the real numbers, the complex numbers, the quaternions, and the octonions. This paper is part of a sequence, including [DDD] and [DDDD], that explores some detailed algebraic properties of these algebras.

Classically, the first four Cayley-Dickson algebras, i.e., $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, are viewed as well-behaved, while the larger Cayley-Dickson algebras are considered to be pathological. There are several different ways of making this distinction. One difference is that the first four algebras do not possess zero-divisors, while the higher algebras do have zero-divisors. One of our primary long-term goals is to understand the zero-divisors in as much detail as possible; the papers [DDD] and [DDDD] more directly address this question. Our motivation for studying zero-divisors is the potential for useful applications in topology; see [Co] for more details.

A different but related important property of the first four Cayley-Dickson algebras is that they are *alternative*. This means that $a \cdot ax = a^2x$ for all a and all x. This is obvious for the associative algebras $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. It is also true for the octonions. One important consequence of this fact is that it allows for the construction of the projective line and plane over $\mathbb{O}$ [B].

Alternativity fails in the higher Cayley-Dickson algebras; there exist a and x such that $a \cdot ax$ does not equal a^2x. Because alternativity is so fundamental to the lower Cayley-Dickson algebras, it makes sense to explore exactly how alternativity fails.

For various technical reasons that will be apparent later, it turns out to be inconvenient to consider the operator L_a^2, where L_a is left multiplication by a. Rather, it is preferable to study the operator $M_a = \frac{1}{|a|^2} L_{a^*} L_a$, where $|\, a \,|$ is the norm of a and a^* is the conjugate of a. We will show that M_a is diagonalizable over $\mathbb{R}$. Moreover, its eigenvalues are non-negative.

Thus we are led to consider the eigentheory of M_a. Given a, we desire to describe the eigenvalues and eigenspaces of M_a in as much detail as possible.

This approach to Cayley-Dickson algebras was begun in [MG]. However, for completeness, we have reproved everything that we need here.

Although the elegance of our results about the eigentheory of M_a speaks for itself, we give a few reasons why this viewpoint on Cayley-Dickson algebras is useful. First, it is possible to completely classify all subalgebras of the 16-dimensional Cayley-Dickson algebra. See [CD] for a different proof of this subalgebra classification. On the subject of subalgebras of Cayley-Dickson algebras, the article [Ca] is worth noting.

Second, eigentheory supplies one possible solution to the *cancellation problem*. Namely, given a and b, is it possible to find x such that $ax = b$? The problem is a technical but essential idea in [DDDD, Section 6].

With alternativity, one can multiply this equation by a^* on the left and compute that $|a|^2 x = a^*b$. Since $|a|^2$ is a non-zero real number for any non-zero a, this determines x explicitly.

Now we explain how to solve the equation $ax = b$ without alternativity. Write $x = \sum x_i$ and $b = \sum b_i$, where b_i and x_i belong to the λ_i-eigenspace of M_a. Multiply on the left by a^* to obtain $a^* \cdot ax = a^*b$, which can be rewritten as $|a|^2 \sum \lambda_i x_i = \sum a^* b_i$. As long as none of the eigenvalues λ_i are zero, each x_i equals $\frac{1}{\lambda_i |a|^2} a^* b_i$, and therefore x can be recovered. We expect problems with cancellation when one of the eigenvalues is zero; this corresponds to the fact that if a is a zero-divisor, then the cancellation problem might have no solution or might have non-unique solutions.

We would like to draw the reader's attention to a number of open questions in Section 9.

1.1. **Conventions.** This paper is not intended to stand independently. In particular, we rely heavily on background from [DDD]. Section 2 reviews the main points that we will use.

2. Cayley-Dickson algebras

The **Cayley-Dickson algebras** are a sequence of non-associative $\mathbb{R}$-algebras with involution. See [DDD] for a full explanation of the basic properties of Cayley-Dickson algebras.

These algebras are defined inductively. We start by defining $\boldsymbol{A_0}$ to be $\mathbb{R}$. Given A_{n-1}, the algebra $\boldsymbol{A_n}$ is defined additively to be $A_{n-1} \times A_{n-1}$. Conjugation in A_n is defined by

$$(a, b)^* = (a^*, -b),$$

and multiplication is defined by

$$(a, b)(c, d) = (ac - d^*b, da + bc^*).$$

One can verify directly from the definitions that A_1 is isomorphic to $\mathbb{C}$; A_2 is isomorphic to $\mathbb{H}$; and A_3 is isomorphic to the octonions $\mathbb{O}$. The next algebra A_4 is 16-dimensional; it is sometimes called the *hexadecanions* or the *sedenions*.

We implicitly view A_{n-1} as the subalgebra $A_{n-1} \times 0$ of A_n.

2.1. **Complex structure.** The element $\boldsymbol{i_n} = (0, 1)$ of A_n enjoys many special properties. One of the primary themes of our long-term project is to fully exploit these special properties.

Let $\mathbb{C}_{\boldsymbol{n}}$ be the $\mathbb{R}$-linear span of $1 = (1, 0)$ and i_n. It is a subalgebra of A_n that is isomorphic to $\mathbb{C}$. An easy consequence of [DDD, Lem. 5.5] is that $a^*(ai_n) = (a^*a)i_n$ for all a in A_n.

Lemma 2.2 (DDD, Prop. 5.3)**.** *Under left multiplication, A_n is a $\mathbb{C}_n$-vector space. In particular, if α and β belong to $\mathbb{C}_n$ and x belongs to A_n, then $\alpha(\beta x) = (\alpha\beta)x$.*

As a consequence, the expression $\alpha\beta x$ is unambiguous; we will usually simplify notation in this way.

The **real part $\mathbf{Re}(\boldsymbol{x})$** of an element x of A_n is defined to be $\frac{1}{2}(x+x^*)$, while the **imaginary part $\mathbf{Im}(\boldsymbol{x})$** is defined to be $x - \mathrm{Re}(x)$.

The algebra A_n becomes a positive-definite real inner product space when we define $\langle \boldsymbol{a}, \boldsymbol{b}\rangle_{\mathbb{R}} = \mathrm{Re}(ab^*)$ [DDD, Prop. 3.2]. The real inner product allows us to define a positive-definite Hermitian inner product on A_n by setting $\langle \boldsymbol{a}, \boldsymbol{b}\rangle_{\mathbb{C}}$ to be the orthogonal projection of ab^* onto the subspace $\mathbb{C}_n$ of A_n [DDD, Prop. 6.3]. We say that two elements a and b are **$\mathbb{C}$-orthogonal** if $\langle a, b\rangle_{\mathbb{C}} = 0$.

For any a in A_n, let $\boldsymbol{L_a}$ and $\boldsymbol{R_a}$ be the linear maps $A_n \to A_n$ given by left and right multiplication by a respectively.

Lemma 2.3 (M1, Lem. 1.3, DDD, Lem. 3.4)**.** *Let a be any element of A_n. With respect to the real inner product on A_n, the adjoint of L_a is L_{a^*}, and the adjoint of R_a is R_{a^*}.*

We will need the following slightly technical result.

Lemma 2.4. *Let x and y be elements of A_n such that y is imaginary. Then x and xy are orthogonal.*

Proof. We wish to show that $\langle x, xy\rangle_{\mathbb{R}}$ is zero. By Lemma 2.3, this equals $\langle x^*x, y\rangle_{\mathbb{R}}$, which is zero because x^*x is real while y is imaginary. □

We will frequently consider the subspace $\mathbb{C}_{\boldsymbol{n}}^{\perp}$ of A_n; it is the orthogonal complement of $\mathbb{C}_n$ (with respect either to the real or to the Hermitian inner product). Note that $\mathbb{C}_n^{\perp}$ is a $\mathbb{C}_n$-vector space; in other words, if a belongs to $\mathbb{C}_n^{\perp}$ and α belongs to $\mathbb{C}_n$, then αa also belongs to $\mathbb{C}_n^{\perp}$ [DDD, Lem. 3.8].

Lemma 2.5 (DDD, Lem. 6.4 and 6.5)**.** *If a belongs to $\mathbb{C}_n^{\perp}$, then L_a is $\mathbb{C}_n$-conjugate-linear in the sense that $L_a(\alpha x) = \alpha^* L_a(x)$ for any x in A_n and any α in $\mathbb{C}_n$. Moreover, L_a is anti-Hermitian in the sense that $\langle L_a x, y\rangle_{\mathbb{C}} = -\langle x, L_a y\rangle_{\mathbb{C}}^*$.*

Similar results hold for R_a. See also [MG, Lem. 2.3] for a different version of the claim about conjugate-linearity.

The conjugate-linearity of L_a is fundamental to many later calculations. To emphasize this point, we provide a few exercises.

Exercise 2.6. Suppose that a and b belong to $\mathbb{C}_n^{\perp}$, while α belongs to $\mathbb{C}_n$. Show that:

(1) $\alpha a = a\alpha^*$.
(2) $a \cdot \alpha b = \alpha^* \cdot ab$.
(3) $\alpha a \cdot b = ab \cdot \alpha$.

Exercise 2.7. Let a and b belong to $\mathbb{C}_n^{\perp}$, and let α and β belong to $\mathbb{C}_n$. Suppose also that a and b are $\mathbb{C}$-orthogonal. Prove that

$$\alpha a \cdot \beta b = \alpha^* \beta^* \cdot ab.$$

In this limited sense, multiplication is bi-conjugate-linear.

2.8. **Norms.** Norms of elements in A_n are defined with respect to either the real or Hermitian inner product: $|a| = \sqrt{\langle a, a\rangle_{\mathbb{R}}} = \sqrt{\langle a, a\rangle_{\mathbb{C}}} = \sqrt{aa^*}$; this makes sense because aa^* is always a non-negative real number [DDD, Lem. 3.6]. Note also that $|a| = |a^*|$ for all a.

Lemma 2.9. *If a belongs to $\mathbb{C}_n^\perp$ and α belongs to $\mathbb{C}_n$, then $|\alpha a| = |\alpha||a|$.*

Proof. By Lemmas 2.2 and 2.5,

$$\langle \alpha a, \alpha a\rangle_{\mathbb{C}} = \langle \alpha^* \alpha a, a\rangle_{\mathbb{C}} = \langle |\alpha|^2 a, a\rangle_{\mathbb{C}} = |\alpha|^2 |a|^2 .$$

□

Lemma 2.10. *For any x and y in A_n, $|xy| = |yx|$.*

Proof. If x and y are imaginary, then $yx = y^* x^* = (xy)^*$ because $x = -x^*$, $y = -y^*$, and conjugation respects multiplication after reversing order. By [DDD, Lem. 3.6], the norms of xy and $(xy)^*$ are equal.

In general, write $x = r + x'$ and $y = s + y'$, where r and s are real and x' and y' are imaginary. Then $xy = rs + sx' + ry' + x'y'$ and $yx = rs + sx' + ry' + y'x'$ because real scalars are central. Therefore, $(yx)^*$ equals $rs - sx' - ry' + x'y'$. To show that the norms of xy and $(yx)^*$ are equal, it suffices to show that $rs + x'y'$ and $sx' + ry'$ are orthogonal.

Note that rs is orthogonal to $sx' + ry'$ because the first expression is real while the second one is imaginary. On the other hand, $x'y'$ is orthogonal to $sx' + ry'$ by Lemma 2.4. □

2.11. **Standard basis.** The algebra A_n is equipped with an inductively defined **standard $\mathbb{R}$-basis** [DDD, Defn. 2.10]. The standard $\mathbb{R}$-basis is orthonormal.

Definition 2.12. *An element a of A_n is **alternative** if $a \cdot ax = a^2 x$ for all x. An algebra is said to be alternative if all of its elements are alternative.*

The Cayley-Dickson algebra A_n is alternative if and only if $n \leq 3$.

Lemma 2.13 (DDD, Lem. 4.4)**.** *Standard basis elements are alternative.*

2.14. **Subalgebras.** A subalgebra of A_n is an $\mathbb{R}$-linear subspace containing 1 that is closed under both multiplication and conjugation.

Definition 2.15. *For any elements $a_1, a_2, \ldots, a_k$ in A_n, let $\langle\langle \boldsymbol{a_1, a_2, \ldots, a_k}\rangle\rangle$ denote the smallest subalgebra of A_n that contains the elements $a_1, a_2, \ldots, a_k$.*

We will usually apply this construction to two elements a and i_n. Unless a belongs to $\mathbb{C}_n$, the subalgebra $\langle\langle a, i_n\rangle\rangle$ is always isomorphic to the quaternions with additive basis consisting of 1, a, i_n, and $i_n a$ [DDD, Lem. 5.6].

Because of non-associativity, some properties of generators of Cayley-Dickson algebras are counter-intuitive. For example, the algebra A_3 is generated by three elements but not by any two elements. On the other hand, A_4 is generated by a generic pair of elements.

2.16. **The octonions.** We recall some properties of A_3 and establish some notation.

In A_3, we write $\boldsymbol{i} = i_1$, $\boldsymbol{j} = i_2$, $\boldsymbol{k} = ij$, and $\boldsymbol{t} = i_3$ because it makes the notation less cumbersome. The standard basis for A_3 is

$$\{1, i, j, k, t, it, jt, kt\}.$$

The automorphism group of A_3 is the 14-dimensional sporadic Lie group G_2 [B, Sec. 4.1] [DDD, Sec. 7]. It acts transitively on the imaginary elements of length 1. In other words, up to automorphism, all imaginary unit vectors are the same. In fact, $\mathrm{Aut}(A_3)$ acts transitively on ordered pairs of orthogonal imaginary elements of unit length. Even better, $\mathrm{Aut}(A_3)$ acts transitively on ordered triples (x, y, z) of pairwise orthogonal imaginary elements of unit length such that z is also orthogonal to xy.

The subalgebra $\mathbb{C}_3$ is additively generated by 1 and t. However, up to automorphism, we may assume that $\mathbb{C}_3$ is generated by 1 together with any non-zero imaginary element. Similarly, up to automorphism, we may assume that any imaginary element of A_3 is orthogonal to $\mathbb{C}_3$. Such assumptions may not be made in A_n for $n \geq 4$ because the automorphism group of A_n does not act transitively [Br] [ES].

3. Eigentheory

Definition 3.1. *Let a be a non-zero element of A_n. Define $\boldsymbol{M_a}$ to be the $\mathbb{R}$-linear map $\frac{1}{|a|^2} L_{a^*} L_a$. The **eigenvalues of a** are the eigenvalues of M_a. Similarly, the eigenvectors of a are the eigenvectors of M_a. Let $\mathbf{Eig}_{\boldsymbol{\lambda}}\boldsymbol{(a)}$ be the λ-eigenspace of a.*

For any real scalar r, the eigenvalues and eigenvectors of ra are the same as those of a. Therefore, we will assume that $|\, a \,|= 1$ whenever it makes our results easier to state.

Remark 3.2. If a is imaginary, then $a^* = -a$. In this case, x is a λ-eigenvector of a if and only if $a \cdot ax = -\lambda \,|\, a \,|^2\, x$.

Lemma 3.3. *For any a in A_n, M_a equals M_{a^*}.*

Proof. Because $|\, a \,|=|\, a^* \,|$, the claim is that $a^* \cdot ax = a \cdot a^* x$ for all a and x in A_n. To check this, write $a = r + a'$ where r is real and a' is imaginary. Compute directly that

$$(r + a') \cdot (r - a')x = (r - a') \cdot (r + a')x$$

for all x in A_n. □

Remark 3.4. To make sense of the notation in the following proposition, note that any unit vector in A_n can be written in the form $a \cos\theta + \beta \sin\theta$, where a and β are both unit vectors with a in $\mathbb{C}_n^\perp$ and β in $\mathbb{C}_n$. Generically, a and β are unique up to multiplication by -1, and θ is unique up to the obvious redundancies of trigonometry.

Lemma 3.5. *Let a be a unit vector in $\mathbb{C}_n^\perp$, and let β be a unit vector in $\mathbb{C}_n$. Then $M_{a\cos\theta+\beta\sin\theta}$ equals $I \sin^2\theta + M_a \cos^2\theta$.*

Proof. First note that the conjugate of $a \cos\theta + \beta \sin\theta$ is $-a \cos\theta + \beta^* \sin\theta$. Distribute to compute that

$$(-a \cos\theta + \beta^* \sin\theta) \cdot (a \cos\theta + \beta \sin\theta)x =$$
$$\beta^*\beta x \sin^2\theta - a \cdot \beta x \cos\theta \sin\theta + \beta^* \cdot ax \cos\theta \sin\theta - a \cdot ax \cos^2\theta.$$

Using that $\beta^*\beta =|\, \beta \,|^2$ and that $a \cdot \beta x = \beta^* \cdot ax$ by Lemma 2.5, this simplifies to

$$x \sin^2\theta + a^* \cdot ax \cos^2\theta.$$

□

Lemma 3.6. *For any a in A_n, the map M_a is $\mathbb{C}_n$-linear. In particular, every eigenspace of a is a $\mathbb{C}_n$-vector space.*

Proof. We may assume that a is a unit vector. Lemma 3.5 allows us to assume that a is imaginary. Then Lemma 2.5 says that M_a is the composition of two conjugate-linear maps, which means that it is $\mathbb{C}_n$-linear. □

The next result is a technical lemma that will be used in many of our calculations.

Lemma 3.7. *If a, x, and y belong to A_n, then*

$$\langle L_a x, L_a y\rangle_{\mathbb{R}} = |a|^2 \langle M_a x, y\rangle_{\mathbb{R}} = |a|^2 \langle x, M_a y\rangle_{\mathbb{R}}.$$

Proof. This follows immediately from the adjointness properties of Lemma 2.3. □

Lemma 3.8. *For any a in A_n, the kernels of M_a and L_a are equal. In particular, a is a zero-divisor if and only if 0 is an eigenvalue of a.*

Proof. If $ax = 0$, then $a^* \cdot ax = 0$.

For the other direction, suppose that $a^* \cdot ax = 0$. This implies that $\langle M_a x, x\rangle_{\mathbb{R}}$ equals zero, so Lemma 3.7 implies that $\langle L_a x, L_a x\rangle_{\mathbb{R}}$ equals zero. In other words, $|ax|^2 = 0$, so $ax = 0$. □

Proposition 3.9. *For every a in A_n, M_a is diagonalizable with non-negative eigenvalues. If λ_1 and λ_2 are distinct eigenvalues of a, then $\mathrm{Eig}_{\lambda_1}(a)$ and $\mathrm{Eig}_{\lambda_2}(a)$ are orthogonal.*

Proof. Recall from Lemma 2.3 that the adjoint of L_a is L_{a^*}. Therefore, $L_{a^*}L_a$ is symmetric; this shows that M_a is also symmetric. The fundamental theorem of symmetric matrices says that M_a is diagonalizable. The orthogonality of the eigenspaces is a standard property of symmetric matrices.

To show that all of the eigenvalues are non-negative, let $M_a x = \lambda x$ with $\lambda \neq 0$ and $x \neq 0$. The value $\langle x, x\rangle_{\mathbb{R}}$ is positive, and it equals

$$\frac{1}{\lambda}\langle \lambda x, x\rangle_{\mathbb{R}} = \frac{1}{\lambda}\langle M_a x, x\rangle_{\mathbb{R}} = \frac{1}{\lambda}\langle L_a x, L_a x\rangle_{\mathbb{R}}$$

by Lemma 3.7. Since $\langle L_a x, L_a x\rangle_{\mathbb{R}}$ is also positive, it follows that λ must be positive. □

In practice, we will only study eigenvalues of elements of A_n that are orthogonal to $\mathbb{C}_n$. The result below explains that if we understand the eigenvalues in this special case, then we understand them all.

Recall from Remark 3.4 that any unit vector in A_n can be written in the form $a\cos\theta + \beta\sin\theta$, where a is a unit vector in $\mathbb{C}_n^{\perp}$ and β is a unit vector in $\mathbb{C}_n$.

Proposition 3.10. *Let a and β be unit vectors in A_n such that a belongs to $\mathbb{C}_n^{\perp}$ and β belongs to $\mathbb{C}_n$. Then*

$$\mathrm{Eig}_{\lambda}(a) = \mathrm{Eig}_{\sin^2\theta + \lambda\cos^2\theta}(a\cos\theta + \beta\sin\theta).$$

In particular, λ is an eigenvalue of a if and only if $\sin^2\theta + \lambda\cos^2\theta$ is an eigenvalue of $a\cos\theta + \beta\sin\theta$.

Proof. This follows immediately from Lemma 3.5, which says that $M_{a\cos\theta+\beta\sin\theta}$ equals $I\sin^2\theta + M_a\cos^2\theta$. □

Remark 3.11. Note that the case $\lambda = 1$ is special in the above proposition, giving that $\mathrm{Eig}_1(a) = \mathrm{Eig}_1(a\cos\theta + \beta\sin\theta)$. In other words, the 1-eigenspace of an element of A_n depends only on its orthogonal projection onto $\mathbb{C}_n^{\perp}$.

Remark 3.12. Let a and β be unit vectors in A_n such that a belongs to $\mathbb{C}_n^\perp$ and β belongs to $\mathbb{C}_n$. Propositions 3.9 and 3.10 show that the eigenvalues of $a\cos\theta+\beta\sin\theta$ are at least $\sin^2\theta$. In particular, if 0 is an eigenvalue of $a\cos\theta + \beta\sin\theta$, then $\sin\theta = 0$. In other words, zero-divisors are always orthogonal to $\mathbb{C}_n$ [M1, Cor. 1.9] [DDD, Lem. 9.5].

Recall from Section 2.14 that $\langle\langle a, i_n\rangle\rangle$ is the subalgebra generated by a and i_n.

Proposition 3.13. *For any a in A_n, $\langle\langle a, i_n\rangle\rangle$ is contained in $\mathrm{Eig}_1(a)$. In particular, 1 is an eigenvalue of every non-zero element of A_n.*

Proof. First note that $\langle\langle a, i_n\rangle\rangle$ is isomorphic to either $\mathbb{C}$ or $\mathbb{H}$; this follows from [DDD, Lem. 5.6]. In either case, it is an associative subalgebra. Therefore, $a^* \cdot ax = a^*a \cdot x = |a|^2 x$ for any x in $\langle\langle a, i_n\rangle\rangle$. □

Proposition 3.14. *Let a and β be non-zero vectors in A_n such that β belongs to $\mathbb{C}_n$. Then $\mathrm{Eig}_\lambda(a) = \mathrm{Eig}_\lambda(\beta a)$ for any λ. In particular, the eigenvalues of a and βa are the same.*

See also [MG, Cor. 3.6] for a related result in different notation.

Proof. We may assume that a and β both have norm 1.

Proposition 3.10 implies that the result holds for all a if it holds for a in $\mathbb{C}_n^\perp$. Therefore, we may assume that a is orthogonal to $\mathbb{C}_n$.

If x belongs to $\langle\langle a, i_n\rangle\rangle$, then Proposition 3.13 says that x belongs to $\mathrm{Eig}_1(a)$. However, $\langle\langle a, i_n\rangle\rangle$ and $\langle\langle \beta a, i_n\rangle\rangle$ are equal, so Proposition 3.13 again says that x belongs to $\mathrm{Eig}_1(\beta a)$.

Since the eigenspaces of a are orthogonal by Proposition 3.9, we may now assume that x is orthogonal to $\langle\langle a, i_n\rangle\rangle$. In other words, x is $\mathbb{C}$-orthogonal to both 1 and a. Under this hypothesis, compute

$$(\beta a)^* \cdot (\beta a)x = \beta^*\beta(a^* \cdot ax) = |\beta|^2\, a^* \cdot ax = a^* \cdot ax$$

using Lemma 2.5. In this computation, we need that ax is orthogonal to $\mathbb{C}_n$; this is equivalent to the assumption that a and x are $\mathbb{C}$-orthogonal.

Recalling from Lemma 2.9 that $|\beta a| = |\beta||a| = 1$, this calculation shows that x belongs to $\mathrm{Eig}_\lambda(a)$ if and only if it belongs to $\mathrm{Eig}_\lambda(\beta a)$. □

Lemma 3.15. *For all x and y in A_n, $\mathrm{tr}(L_{x^*}L_y)$ equals $2^n\langle x, y\rangle_\mathbb{R}$.*

Proof. Let $b(x, y) = \mathrm{tr}(L_{x^*}L_y)$. Since both $b(x, y)$ and $2^n\langle x, y\rangle_\mathbb{R}$ are bilinear in x and y, it suffices to assume that x and y belong to the standard basis described in Section 2.11.

First suppose that $x = y$. Because $|x| = 1$ and x is alternative by Lemma 2.13,

$$b(x, x) = \mathrm{tr}(L_{x^*}L_x) = \mathrm{tr}(I) = 2^n.$$

It now suffices to assume that x and y are distinct standard basis elements and to show that $b(x, y) = 0$. We want to compute

$$b(x, y) = \Sigma_z\langle z, L_{x^*}L_y z\rangle_\mathbb{R},$$

where z ranges over the standard basis. For each z, $\langle z, L_{x^*}L_y z\rangle_\mathbb{R}$ equals $\langle xz\cdot z^*, y\rangle_\mathbb{R}$, by two applications of Lemma 2.3. Using that z is alternative by Lemma 2.13, we get $\langle x \cdot zz^*, y\rangle_\mathbb{R}$. Now zz^* is a real scalar, and $\langle x, y\rangle_\mathbb{R}$ is zero by assumption. Thus every term in the sum for $b(x, y)$ is zero, so $b(x, y) = 0$. □

Proposition 3.16. *For any a in A_n, the sum of the eigenvalues of a is equal to 2^n.*

Proof. This follows immediately from Lemma 3.15 because the trace of a diagonalizable operator equals the sum of its eigenvalues. □

Lemma 3.17. *For any a in A_n, the map L_a takes $\mathrm{Eig}_\lambda(a)$ into $\mathrm{Eig}_\lambda(a)$. If λ is non-zero, then L_a restricts to an automorphism of $\mathrm{Eig}_\lambda(a)$.*

Proof. Let x belong to $\mathrm{Eig}_\lambda(a)$. Using that $M_a = M_{a^*}$ from Lemma 3.3, compute that

$$M_a(ax) = M_{a^*}(ax) = \frac{1}{|a|^2} a \cdot a^*(ax) = a \cdot M_a x = \lambda a x.$$

This shows that ax also belongs to $\mathrm{Eig}_\lambda(a)$.

For the second claim, simply note that $L_{a^*}L_a$ is scalar multiplication by $\lambda\,|a|^2$ on $\mathrm{Eig}_\lambda(a)$. Thus the inverse to L_a is $\frac{1}{\lambda|a|^2}L_{a^*}$. □

Remark 3.18. For $\lambda \neq 0$, the restriction $L_a : \mathrm{Eig}_\lambda(a) \to \mathrm{Eig}_\lambda(a)$ is a similarity in the sense that it is an isometry up to scaling. This follows from Lemma 3.7.

Also, an immediate consequence of Lemma 3.17 is that the image of L_a is equal to the orthogonal complement of $\mathrm{Eig}_0(a)$. This fact is used in [DDDD].

Proposition 3.19. *Let $n \geq 2$. For any a in A_n, every eigenspace of a is even-dimensional over $\mathbb{C}_n$. In particular, the real dimension of any eigenspace is a multiple of 4.*

See also [MG, Thm. 4.6] for the second claim.

Proof. By Proposition 3.10, we may assume that a is orthogonal to $\mathbb{C}_n$. Let λ be an eigenvalue of a.

Recall from Lemma 2.5 that L_a is a conjugate-linear anti-Hermitian map. If λ is non-zero, then Lemma 3.17 says that L_a restricts to an automorphism of $\mathrm{Eig}_\lambda(a)$. By [DDD, Lem. 6.6], conjugate-linear anti-Hermitian automorphisms exist only on even-dimensional $\mathbb{C}$-vector spaces.

Now consider $\lambda = 0$. The $\mathbb{C}_n$-dimension of $\mathrm{Eig}_0(a)$ is equal to 2^{n-1} minus the dimensions of the other eigenspaces. By the previous paragraph and the fact that 2^{n-1} is even, it follows that $\mathrm{Eig}_0(a)$ is also even-dimensional. □

The previous proposition, together with Propositions 3.9 and 3.16, shows that if a belongs to A_n, then the eigenvalues of a are at most 2^{n-2}. However, this bound is not sharp. Later in Corollary 4.7 we will prove a stronger result.

Proposition 3.20. *Let a belong to A_n, and let $\lambda \geq 0$. Then x belongs to $\mathrm{Eig}_\lambda(a)$ if and only if $|ax| = \sqrt{\lambda}\,|a||x|$ and $|M_a x| = \lambda\,|x|$.*

The above proposition is a surprisingly strong result. We know that $L_{a^*}L_a$ scales an element of $\mathrm{Eig}_\lambda(a)$ by $\lambda\,|a|^2$. The proposition makes the non-obvious claim that this scaling occurs in two geometrically equal stages for the two maps L_{a^*} and L_a. Moreover, it says that as long as the norms of $L_a x$ and $M_a x$ are correct, then the direction of $M_a x$ takes care of itself. In practice, it is a very useful simplification not to have to worry about the direction of $M_a x$. One part of Proposition 3.20 is proved in [MG, Prop. 4.20].

Proof. First suppose that x belongs to $\mathrm{Eig}_\lambda(a)$. The second desired equality follows immediately. For the first equality, use Lemma 3.7 to compute that

$$|ax|^2 = |a|^2 \langle M_a x, x\rangle_{\mathbb{R}} = \lambda |a|^2 |x|^2 .$$

Now take square roots. This finishes one direction.

For the other direction, note that x belongs to $\mathrm{Eig}_\lambda(a)$ if and only if the norm of $M_a x - \lambda x$ is zero. Using the formulas in the proposition and Lemma 3.7, compute that

$$\begin{aligned} \langle M_a x - \lambda x, M_a x - \lambda x\rangle_{\mathbb{R}} &= \\ |M_a x|^2 + \lambda^2 |x|^2 - 2\lambda \langle M_a x, x\rangle_{\mathbb{R}} &= \\ \lambda^2 |x|^2 + \lambda^2 |x|^2 - 2\lambda \frac{|ax|^2}{|a|^2} &= \\ 2\lambda^2 |x|^2 - 2\lambda \frac{\lambda |a|^2 |x|^2}{|a|^2} &= 0. \end{aligned}$$

□

4. Maximum and minimum eigenvalues

Definition 4.1. *For a in A_n, let $\boldsymbol{\lambda_a^-}$ denote the minimum eigenvalue of a, and let $\boldsymbol{\lambda_a^+}$ denote the maximum eigenvalue of a.*

Recall from Proposition 3.13 that 1 is always an eigenvalue of a if a is non-zero. Therefore, λ_a^+ is always at least 1, λ_a^- is at most 1, and $\lambda_a^- = \lambda_a^+$ if and only if a is alternative.

Definition 4.2. *Let a and x belong to A_n. The* ***eigendecomposition of x with respect to a*** *is the sum $x = x_1 + \cdots + x_k$ where each x_i is an eigenvector of a with eigenvalue λ_i such that the λ_i are distinct.*

Note that the eigendecomposition of x with respect to a is unique up to reordering. Note also from Proposition 3.9 that the eigendecomposition of x is an orthogonal decomposition in the sense that x_i and x_j are orthogonal for distinct i and j.

Lemma 4.3. *Let a and x belong to A_n, and let $x_1 + \cdots + x_k$ be the eigendecomposition of x with respect to a. Then $ax_1 + \cdots + ax_k$ is the eigendecomposition of ax with respect to a, except that the term ax_i must be removed if x_i belongs to* $\mathrm{Eig}_0(a)$.

Proof. This follows immediately from Lemma 3.17. □

Lemma 4.4. *Let a belong to A_n. For any x in A_n,*

$$\sqrt{\lambda_a^-}\, |a||x| \leq |ax| \leq \sqrt{\lambda_a^+}\, |a||x| .$$

Proof. Let $x = x_1 + \cdots + x_k$ be the eigendecomposition of x with respect to a, so $ax = ax_1 + \cdots + ax_k$ is the eigendecomposition of ax with respect to a by Lemma 4.3 (except possibly that one term must be dropped). Let λ_i be the eigenvalue of x_i with respect to a. Using Proposition 3.20 and using that eigendecompositions are orthogonal decompositions by Proposition 3.9, we have

$$\begin{aligned}|ax|^2 = |ax_1|^2 + \cdots + |ax_k|^2 &= \lambda_1 \, |a|^2 |x_1|^2 + \cdots + \lambda_k \, |a|^2 |x_k|^2 \\ &\leq \lambda_a^+ \, |a|^2 \left(|x_1|^2 + \cdots + |x_k|^2 \right) \\ &= \lambda_a^+ \, |a|^2 |x|^2 .\end{aligned}$$

The inequality involving λ_a^- is derived similarly. □

Proposition 4.5. *Let a belong to A_n. Then x belongs to $\mathrm{Eig}_{\lambda_a^+}(a)$ if and only if $|ax| = \sqrt{\lambda_a^+} \, |a||x|$. Also, x belongs to $\mathrm{Eig}_{\lambda_a^-}(a)$ if and only if $|ax| = \sqrt{\lambda_a^-} \, |a||x|$.*

The reader should compare this result to Proposition 3.20. We are claiming that for the minimum and maximum eigenvalues, the second condition is redundant.

Proof. We give the proof of the first statement; the proof of the second statement is the same.

One direction is an immediate consequence of Proposition 3.20. For the other direction, suppose that $|ax| = \sqrt{\lambda_a^+} \, |a||x|$. Let $x = x_1 + \cdots + x_k$ be the eigendecomposition of x with respect to a, so $ax = ax_1 + \cdots + ax_k$ is the eigendecomposition of ax with respect to a (except possibly that one term must be dropped). Let λ_i be the eigenvalue of x_i with respect to a.

By Proposition 3.20, we have

$$\begin{aligned}\lambda_a^+ \, |a|^2 \left(|x_1|^2 + \cdots + |x_k|^2 \right) = \lambda_a^+ \, |a|^2 |x|^2 &= |ax|^2 \\ &= |ax_1|^2 + \cdots + |ax_k|^2 \\ &= |a|^2 \left(\lambda_1 \, |x_1|^2 + \cdots + \lambda_k \, |x_k|^2 \right).\end{aligned}$$

Rearrange this equality to get

$$(\lambda_a^+ - \lambda_1) \, |x_1|^2 + \cdots + (\lambda_a^+ - \lambda_k) \, |x_k|^2 = 0.$$

The coefficients $\lambda_a^+ - \lambda_i$ are all positive except for the one value of j for which $\lambda_j = \lambda_a^+$. It follows that $x_i = 0$ for $i \neq j$ and hence $x = x_j$, so x belongs to $\mathrm{Eig}_{\lambda_a^+}(a)$. □

Proposition 4.6. *Let $a = (b, c)$ belong to A_n, where b and c are elements of A_{n-1}. Then $\lambda_a^+ \leq 2 \max\{\lambda_b^+, \lambda_c^+\}$.*

Proof. Let $x = (y, z)$ belong to A_n, and consider the product

$$ax = (b, c)(y, z) = (by - z^* c, cy^* + zb).$$

The triangle inequality and Lemma 2.10 says that

$$\begin{aligned}|ax| &= \sqrt{|by - z^* c|^2 + |cy^* + zb|^2} \\ &\leq \sqrt{|by|^2 + |cz^*|^2 + |cy^*|^2 + |bz|^2 + 2\,|by||cz^*| + 2\,|cy^*||bz|}.\end{aligned}$$

Writing $B = |b|$, $C = |c|$, $Y = |y|$, and $Z = |z|$, repeated use of Lemma 4.4 (and the facts that $|z^*| = |z|$ and $|y^*| = |y|$) gives the inequality

$$|ax| \leq \sqrt{\lambda_b^+ B^2 Y^2 + \lambda_c^+ C^2 Z^2 + \lambda_c^+ C^2 Y^2 + \lambda_b^+ B^2 Z^2 + 4\sqrt{\lambda_b^+ \lambda_c^+} BCYZ}.$$

Replacing λ_b^+, λ_c^+, and $\sqrt{\lambda_b^+ \lambda_c^+}$ with $\max\{\lambda_b^+, \lambda_c^+\}$, we obtain

$$|ax| \leq \sqrt{\max\{\lambda_b^+, \lambda_c^+\}} \sqrt{B^2 Y^2 + C^2 Z^2 + C^2 Y^2 + B^2 Z^2 + 4BCYZ}.$$

Next, use the inequalities $2BC \leq B^2 + C^2$ and $2YZ \leq Y^2 + Z^2$ to get

$$|ax| \leq \sqrt{\max\{\lambda_b^+, \lambda_c^+\}}\sqrt{2(B^2+C^2)(Y^2+Z^2)} = \sqrt{2\max\{\lambda_b^+, \lambda_c^+\}}\,|a||x|\,.$$

Since this inequality holds for all x, we conclude by Proposition 4.5 that $\lambda_a^+ \leq 2\max\{\lambda_b^+, \lambda_c^+\}$. □

Corollary 4.7. *Let $n \geq 3$. If a belongs to A_n, then all eigenvalues of a are in the interval $[0, 2^{n-3}]$.*

Proof. The proof is by induction, using Propositions 3.9 and 4.6. The base case is $n = 3$. Recall that A_3 is alternative, so 1 is the only eigenvalue of any a in A_3. □

Remark 4.8. Corollary 4.7 is sharp in the following sense. For $n \geq 4$, every real number in the interval $[0, 2^{n-3}]$ occurs as the eigenvalue of some element of A_n. See Theorem 8.3 for more details.

5. Cross-product

Definition 5.1. *Given a and b in A_n, let the **cross-product $a \times b$** be the imaginary part of ab^*.*

If $\mathbb{R}^3$ is identified with the imaginary part of A_2, then this definition restricts to the usual notion of cross-product in physics. The cross-product has also been previously studied for A_3; see [B, Sec. 4.1] for example. We shall see that cross-products are indispensible in describing eigenvalues and eigenvectors, especially for A_4.

Lemma 5.2. *Let a and b belong to A_n, and let θ be the angle between a and b. If b belongs to $\mathrm{Eig}_1(a)$ or a belongs to $\mathrm{Eig}_1(b)$, then*

$$|a \times b| = |a||b| \sin\theta.$$

Proof. Note that $ab^* = \mathrm{Re}(ab^*) + \mathrm{Im}(ab^*)$ is an orthogonal decomposition of ab^*. Therefore,

$$|\mathrm{Im}(ab^*)|^2 = |ab^*|^2 - |\mathrm{Re}(ab^*)|^2 = |ab^*|^2 - \langle a, b\rangle_{\mathbb{R}}^2.$$

By Proposition 3.20, we know that $|ab^*|^2 = |a|^2|b^*|^2 = |a|^2|b|^2$, so we get that

$$|\mathrm{Im}(ab^*)|^2 = |a|^2|b|^2 - |a|^2|b|^2\cos^2\theta = |a|^2|b|^2\sin^2\theta.$$

□

Lemma 5.3. *Let a and b belong to A_n such that b belongs to $\mathrm{Eig}_1(a)$ or a belongs to $\mathrm{Eig}_1(b)$. Then $|a \times b| \leq \frac{1}{2}(|a|^2 + |b|^2)$. Moreover, $|a \times b| = \frac{1}{2}(|a|^2 + |b|^2)$ if and only if a and b are orthogonal and have the same norm. Also $a \times b = 0$ if and only if a and b are linearly dependent.*

Proof. The inequality follows from Lemma 5.2 together with the simple observation that

$$|a|^2 + |b|^2 \geq 2\,|a||b| \geq 2\,|a||b|\sin\theta.$$

It then follows that $|a \times b| = \frac{1}{2}(|a|^2 + |b|^2)$ if and only if $|a|^2 + |b|^2 = 2\,|a||b|$ and $\sin\theta = 1$. These two conditions occur if and only if $|a| = |b|$ and $\theta = \frac{\pi}{2}$.

Finally, Lemma 5.2 shows that $a \times b = 0$ if and only if $a = 0$, $b = 0$, $\theta = 0$, or $\theta = \pi$. □

Lemma 5.4. *Let a belong to $\mathbb{C}_{n-1}^{\perp}$, and let α and β belong to $\mathbb{C}_{n-1}$. Then*

$$\alpha a \times \beta a = |a|^2 (\alpha \times \beta).$$

Proof. Use Lemma 2.5 to compute that $(\alpha a)(\beta a)^* = |a|^2 \alpha\beta^*$. □

Lemma 5.5. *Let a and b be imaginary elements of A_n. Then $a \times b$ is orthogonal to both a and b.*

Proof. Lemma 2.4 says that ab is orthogonal to both a and b. Also, a and b are orthogonal to $\mathrm{Re}(ab)$ because they are imaginary. Therefore, a and b are orthogonal to $\mathrm{Im}(ab) = ab - \mathrm{Re}(ab)$. □

Even though $a \times b$ is always orthogonal to a and b in the previous lemma, beware that $a \times b$ may equal zero even if a and b are orthogonal.

6. Eigenvalues and basic constructions

Throughout this section, the reader should keep the following ideas in mind. We will consider elements of A_n of the form $(\alpha a, \beta a)$, where a belongs to $\mathbb{C}_{n-1}^{\perp}$ and α and β belong to $\mathbb{C}_{n-1}$. Under these circumstances, Lemma 5.4 applies, and we conclude that $\alpha a \times \beta a$ always belongs to $\mathbb{C}_{n-1}$. Moreover, since $\alpha a \times \beta a$ is imaginary, it is in fact an $\mathbb{R}$-multiple of i_{n-1}. Even more precisely, $\alpha a \times \beta a$ equals $\pm |\alpha \times \beta| i_{n-1}$.

For a in $\mathbb{C}_{n-1}^{\perp}$, recall from Section 2.14 that $\langle\langle a, i_{n-1}\rangle\rangle$ is the subalgebra of A_{n-1} generated by a and i_{n-1}. It is isomorphic to $\mathbb{H}$.

Lemma 6.1. *Let a belong to $\mathbb{C}_{n-1}^{\perp}$, and let α and β belong to $\mathbb{C}_{n-1}$. If x and y are orthogonal to $\langle\langle a, i_{n-1}\rangle\rangle$, then $(\alpha a, \beta a) \cdot (\alpha a, \beta a)(x, y)$ equals*

$$\Big((|\alpha|^2 + |\beta|^2)a \cdot ax + 2(\alpha \times \beta)(a \cdot ay), (|\alpha|^2 + |\beta|^2)a \cdot ay - 2(\alpha \times \beta)(a \cdot ax)\Big).$$

Proof. Compute using Lemma 2.5. This is a generalized version of the computations in [DDD, Sec. 10]. □

Proposition 6.2. *Let a belong to $\mathbb{C}_{n-1}^{\perp}$, and let α and β belong to $\mathbb{C}_{n-1}$ such that $|a|$ and $|\alpha|^2 + |\beta|^2$ both equal 1 (so that $(\alpha a, \beta a)$ is a unit vector). Suppose that $\alpha \times \beta$ is non-zero (i.e., α and β are $\mathbb{R}$-linearly independent). Let $\gamma = \alpha \times \beta / |\alpha \times \beta|$.*

(a) $\langle\langle a, i_{n-1}, i_n\rangle\rangle$ is contained in the 1-eigenspace of $(\alpha a, \beta a)$;
(b) $\{(x, -\gamma x) : x \in \mathrm{Eig}_1(a) \cap \langle\langle a, i_{n-1}\rangle\rangle^{\perp}\}$ is contained in the $(1 + 2 |\alpha \times \beta|)$-eigenspace of $(\alpha a, \beta a)$;
(c) $\{(x, \gamma x) : x \in \mathrm{Eig}_1(a) \cap \langle\langle a, i_{n-1}\rangle\rangle^{\perp}\}$ is contained in the $(1 - 2 |\alpha \times \beta|)$-eigenspace of $(\alpha a, \beta a)$;
(d) $\{(x, -\gamma x) : x \in \mathrm{Eig}_\lambda(a)\}$ is contained in the $(1 + 2 |\alpha \times \beta|)\lambda$-eigenspace of $(\alpha a, \beta a)$;
(e) $\{(x, \gamma x) : x \in \mathrm{Eig}_\lambda(a)\}$ is contained in the $(1 - 2 |\alpha \times \beta|)\lambda$-eigenspace of $(\alpha a, \beta a)$;

Proof. Note that $\langle\langle a, i_{n-1}, i_n\rangle\rangle$ is an algebra that contains $(\alpha a, \beta a)$ and is isomorphic to the octonions. This establishes part (a) because the octonions are alternative.

Now suppose that x is a λ-eigenvector of a and is orthogonal to $\langle\langle a, i_{n-1}\rangle\rangle$. By Lemma 3.6, $\pm\gamma x$ is also a λ-eigenvector of a and is orthogonal to $\langle\langle a, i_{n-1}\rangle\rangle$. Hence Lemma 6.1 applies, and we compute that $(\alpha a, \beta a)\cdot(\alpha a, \beta a)(x, \pm\gamma x)$ equals

$$\Big(-\lambda x \mp 2(\alpha\times\beta)\lambda\gamma x, \mp\lambda\gamma x + 2(\alpha\times\beta)\lambda x\Big).$$

Recall that γ is an imaginary unit vector, so $\gamma^2 = -1$. It follows that the above expression equals

$$-\lambda\left(1 \mp 2\,|\,\alpha\times\beta\,|\right)(x, \pm\gamma x).$$

Parts (b) through (e) are direct consequences of this formula. □

Remark 6.3. By counting dimensions, it is straightforward to check that A_n is the direct sum of the subspaces listed in the proposition. Thus, the proposition completely describes the eigentheory of $(\alpha a, \beta a)$. Note also that $\langle\langle a, i_{n-1}, i_n\rangle\rangle$ consists of elements of the form (x, y), where x and y both belong to $\langle\langle a, i_{n-1}\rangle\rangle$. Finally, it is important to keep in mind that γ always equals i_{n-1} or $-i_{n-1}$.

Corollary 6.4. *Let a belong to $\mathbb{C}_{n-1}^{\perp}$, and let α and β belong to $\mathbb{C}_{n-1}$. Suppose that $\alpha\times\beta$ is non-zero (i.e., α and β are $\mathbb{R}$-linearly independent). Every eigenvalue of $(\alpha a, \beta a)$ equals either 1 or is of the form*

$$\left(1 \pm \frac{2\,|\,\alpha\times\beta\,|}{|\,\alpha\,|^2 + |\,\beta\,|^2}\right)\lambda,$$

where λ is an eigenvalue of a.

Proof. Let $N = |\,\alpha\,|^2 + |\,\beta\,|^2$. Note that $|\,(\alpha a, \beta a)\,|^2$ equals $N\,|\,a\,|^2$ by Lemma 2.9.

Now consider the unit vector $\left(\frac{\alpha}{\sqrt{N}}\frac{a}{|a|}, \frac{\beta}{\sqrt{N}}\frac{a}{|a|}\right)$. This vector is an $\mathbb{R}$-multiple of $(\alpha a, \beta a)$, so we just need to compute the eigenvalues of this unit vector.

Apply Proposition 6.2 and conclude that the eigenvalues of $\left(\frac{\alpha}{\sqrt{N}}\frac{a}{|a|}, \frac{\beta}{\sqrt{N}}\frac{a}{|a|}\right)$ are either 1 or of the form $\left(1 \pm 2\left|\frac{\alpha}{\sqrt{N}} \times \frac{\beta}{\sqrt{N}}\right|\right)\lambda$, where λ is an eigenvalue of a. This expression equals $\left(1 \pm \frac{2|\alpha\times\beta|}{N}\right)\lambda$, as desired. □

In practice, the multiplicities of the eigenvalues in Corollary 6.4 can be computed by inspection of Proposition 6.2. However, precise results are difficult to state because of various special cases. For example, $1 \pm \frac{2|\alpha\times\beta|}{|\alpha|^2+|\beta|^2}$ are eigenvalues of $(\alpha a, \beta a)$ only if $\mathrm{Eig}_1(a)$ strictly contains $\langle\langle a, i_{n-1}\rangle\rangle$. Also, it is possible that

$$\left(1 - \frac{2\,|\,\alpha\times\beta\,|}{|\,\alpha\,|^2 + |\,\beta\,|^2}\right)\lambda = \left(1 + \frac{2\,|\,\alpha\times\beta\,|}{|\,\alpha\,|^2 + |\,\beta\,|^2}\right)\mu$$

for distinct eigenvalues λ and μ of a.

Because of part (a) of Proposition 6.2, the dimension of $\mathrm{Eig}_1(\alpha a, \beta a)$ is always at least 8.

Corollary 6.5. *Let a belong to $\mathbb{C}_{n-1}^{\perp}$, and let α and β belong to $\mathbb{C}_{n-1}$ such that $|\,a\,|$ and $|\,\alpha\,|^2 + |\,\beta\,|^2$ both equal 1 (so that $(\alpha a, \beta a)$ is a unit vector). Suppose that $|\,\alpha\times\beta\,| = \frac{1}{2}$ (equivalently, by Lemma 5.3, α and β are orthogonal and have the same norm). Every eigenvalue of $(\alpha a, \beta a)$ equals 0 or 1, or is of the form 2λ where λ is an eigenvalue of a. Moreover,*

(a) the multiplicity of 0 is equal to $2^{n-1} - 4 + \dim\mathrm{Eig}_0(a)$;
(b) the multiplicity of 1 is equal to $8 + \dim\mathrm{Eig}_{\frac{1}{2}}(a)$;

(c) the multiplicity of 2 *is equal to* $\dim \mathrm{Eig}_1(a) - 4$*;*
(d) the multiplicity of any other λ *is equal to* $\dim \mathrm{Eig}_{\frac{\lambda}{2}}(a)$*.*

Beware that if $\mathrm{Eig}_1(a)$ is 4-dimensional (i.e., if $\mathrm{Eig}_1(a)$ equals $\langle\langle a, i_{n-1}\rangle\rangle$), then 2 is not an eigenvalue of $(\alpha a, \beta a)$.

Proof. Note that $1 - 2 \mid \alpha \times \beta \mid = 0$ and $1 + 2 \mid \alpha \times \beta \mid = 2$, so parts (c) and (e) of Proposition 6.2 describe $\mathrm{Eig}_0(\alpha a, \beta a)$. The analysis of the other eigenvalues follows from the other parts of Proposition 6.2. □

We end this section by considering the case when $\alpha \times \beta = 0$; this is excluded in Proposition 6.2, Corollary 6.4, and Corollary 6.5.

Proposition 6.6. *Let* a *belong to* $\mathbb{C}_{n-1}^{\perp}$*, and let* α *and* β *belong to* $\mathbb{C}_{n-1}$*. Suppose that* $\alpha \times \beta = 0$ *(equivalently,* α *and* β *are linearly dependent). Then* $\mathrm{Eig}_\lambda(a) \times \mathrm{Eig}_\lambda(a)$ *is contained in* $\mathrm{Eig}_\lambda(\alpha a, \beta a)$*. In particular, the eigenvalues of* $(\alpha a, \beta a)$ *are the same as the eigenvalues of* a*, but the multiplicities are doubled.*

Proof. This follows immediately from the formula in Lemma 6.1. □

7. Eigentheory of A_4

In this section we will completely describe the eigentheory of every element of $\mathbb{C}_4^{\perp}$. The eigentheory of an arbitrary element of A_4 can then be described with Proposition 3.10.

Proposition 7.1. *Let* a *be an imaginary element of* A_3*. Then* $(a\cos\theta, a\sin\theta)$ *is alternative in* A_4*.*

In other words, 1 is the only eigenvalue of (a, b) if a and b are imaginary and linearly dependent elements of A_3.

Proof. As explained in Section 2.16, we may assume that a is orthogonal to $\mathbb{C}_3$. Using that A_3 is alternative, the result is a special case of Proposition 6.6. □

Having dispensed with the linearly dependent case, we will now focus our attention on elements (a, b) of A_4 such that a and b are imaginary and linearly independent.

Theorem 7.2. *Let* a *and* b *be imaginary linearly independent elements of* A_3 *such that* $\mid a \mid^2 + \mid b \mid^2 = 1$ *(so that* (a, b) *is a unit vector in* A_4*). Set* $c = a \times b / \mid a \times b \mid$*. Then:*
(a) $\langle\langle a, b\rangle\rangle \times \langle\langle a, b\rangle\rangle$ *is contained in the* 1*-eigenspace of* (a, b)*;*
(b) $\{(x, -cx) \mid x \in \langle\langle a, b\rangle\rangle^{\perp}\}$ *is contained in the* $(1 + 2 \mid a \times b \mid)$*-eigenspace of* (a, b)*;*
(c) $\{(x, cx) \mid x \in \langle\langle a, b\rangle\rangle^{\perp}\}$ *is contained in the* $(1 - 2 \mid a \times b \mid)$*-eigenspace of* (a, b)*.*

By counting dimensions, it is straightforward to check that A_4 is the direct sum of the subspaces listed in the theorem. Thus, the theorem completely describes the eigentheory of (a, b).

Note that the definition of c makes sense because $a \times b$ is always non-zero when a and b are imaginary and linearly independent.

Proof. The element $a \times b$ is a non-zero imaginary element of A_3. As explained in Section 2.16, we may assume that $a \times b$ is a non-zero scalar multiple of i_3. Then Lemma 5.5 implies that a and b belong to $\mathbb{C}_3^{\perp}$. Now Proposition 6.2 applies. □

Another approach to Theorem 7.2 is to compute directly using octonionic arithmetic that for x in $\langle\langle a, b\rangle\rangle^{\perp}$,

$$(a, b) \cdot (a, b)(x, \pm cx) = -(1 \mp 2\,|\, a \times b\,|)(x, \pm cx).$$

Corollary 7.3. *Let a and b be imaginary linearly independent elements of A_3, and let θ be the angle between a and b. The eigenvalues of (a, b) are*

$$1,\ 1 + \frac{2\,|\,a\,||\,b\,|\,\sin\theta}{|\,a\,|^2 + |\,b\,|^2},\ 1 - \frac{2\,|\,a\,||\,b\,|\,\sin\theta}{|\,a\,|^2 + |\,b\,|^2}.$$

The multiplicities are 8, 4, *and* 4 *respectively.*

Proof. See the proof of Corollary 6.4 to reduce to the case in which (a, b) is a unit vector. Then apply Theorem 7.2. One also needs Lemma 5.2 to compute the norm of the cross-product; note that the hypothesis of this lemma is satisfied because A_3 is alternative. □

Properly interpreted, the corollary is also valid when a and b are linearly dependent. In this case, $\sin\theta = 0$, and all three eigenvalues are equal to 1. This agrees with Proposition 7.1.

Recall that (a, b) is a zero-divisor in A_4 if and only if a and b are orthogonal imaginary elements of A_3 that have the same norm [M1, Cor. 2.14] [DDD, Prop. 12.1].

Proposition 7.4. *Let a and b be orthogonal, imaginary, non-zero elements of A_3 such that $|\,a\,|=|\,b\,|$. Then the eigenvalues of (a, b) in A_4 are* 0, 1, *and* 2 *with multiplicities* 4, 8, *and* 4 *respectively. Moreover,*

(a) $\mathrm{Eig}_0(a, b) = \{(x, -ab \cdot x/\,|\,ab\,|) : x \in \langle\langle a, b\rangle\rangle^{\perp}\}$.
(b) $\mathrm{Eig}_1(a, b) = \langle\langle a, b\rangle\rangle \times \langle\langle a, b\rangle\rangle$.
(c) $\mathrm{Eig}_2(a, b) = \{(x, ab \cdot x/\,|\,ab\,|) : x \in \langle\langle a, b\rangle\rangle^{\perp}\}$.

Proof. This is a special case of Theorem 7.2. Note that ab is already imaginary because a and b are orthogonal; therefore $a \times b = -ab$. □

See [MG, Section 4] for a generic example of the computation in Proposition 7.4.

8. Further Results

We now establish precisely which real numbers occur as eigenvalues in Cayley-Dickson algebras. Recall from [DDD, Prop. 9.10] that if a belongs to A_n, then the dimension of $\mathrm{Eig}_0(a)$ is at most $2^n - 4n + 4$, and this bound is sharp.

Definition 8.1. *A* ***top-dimensional zero-divisor*** *of A_n is a zero-divisor whose 0-eigenspace has dimension $2^n - 4n + 4$.*

Theorem 8.2. *Let a be a top-dimensional zero-divisor in A_n, where $n \geq 3$. Then the eigenvalues of a are* 0 *or 2^k, where $0 \leq k \leq n - 3$. Moreover,*

(a) the multiplicity of 0 *is $2^n - 4n + 4$;*
(b) the multiplicity of 1 *is* 8*;*
(c) the multiplicity of all other eigenvalues is 4.

Proof. The proof is by induction, using Corollary 6.5. The base case $n = 3$ follows from the fact that A_3 is alternative. The base case $n = 4$ follows from Proposition 7.4.

For the induction step, recall from [DDD, Prop. 15.6] (see also [DDDD]) that every unit length top-dimensional zero-divisor of A_n is of the form $\frac{1}{\sqrt{2}}(a, \pm i_{n-1}a)$, where a is a unit length top-dimensional zero-divisor of A_{n-1}. Finally, apply Corollary 6.5. □

Theorem 8.3. *Let $n \geq 4$, and let λ be any real number in the interval $[0, 2^{n-3}]$. There exists an element of A_n that possesses λ as an eigenvalue.*

Proof. From Theorem 8.2, there exists an element a in $\mathbb{C}_n^{\perp}$ that possesses both 0 and 2^{n-3} as an eigenvalue. Proposition 3.10 shows that $a \cos\theta + \sin\theta$ is an element that possesses $\sin^2\theta$ as an eigenvalue. This takes care of the case when $\lambda \leq 1$.

Now suppose that $\lambda \geq 1$. There exists a value of θ for which $\sin^2\theta + 2^{n-3}\cos^2\theta = \lambda$. Proposition 3.10 shows that $a\cos\theta + \sin\theta$ is an element that possesses λ as an eigenvalue. □

9. Some questions for further study

Question 9.1. *Relate the minimum eigenvalue of (b, c) to the minimum eigenvalues of b and c.*

One might hope for an inequality for λ_a^- similar to the inequality given in Proposition 4.6. However, beware that (b, c) can be a zero-divisor, even if neither b nor c are zero-divisors. In other words, $\lambda_{(b,c)}^-$ can equal zero even if λ_b^- and λ_c^- are both non-zero.

Question 9.2. *Let a belong to A_n. Show that the dimension of $\mathrm{Eig}_1(a)$ cannot equal $2^n - 12$ or $2^n - 4$. Show that all other multiples of 4 do occur.*

This guess is supported by computer calculations. It is easy to see that $2^n - 4$ cannot be the dimension of $\mathrm{Eig}_1(a)$. Just use Proposition 3.16.

Computer calculations indicate that the element

$$\big((0, t), (t + it, 1 + i + j)\big)$$

of A_5 possesses 1 as an eigenvalue, and the multiplicity is 4. See Section 2.16 for an explanation of the notation.

Question 9.3. *Fix n. Describe the space of all possible spectra of elements in A_n.*

Results such as Theorem 8.2 suggest that the answer is complicated. We don't even have a guess. A possibly easier question is the following.

Question 9.4. *Fix n. Describe the space of all possible spectra of zero-divisors in A_n.*

Question 9.5. *Study the characteristic polynomial of elements of A_n.*

References

[B] J. C. Baez, *The octonions*, Bull. Amer. Math. Soc. **39** (2002) 145–205.

[Br] R. Brown, *On generalized Cayley-Dickson algebras*, Pacific J. Math. **20** (1967) 415–422.

[Ca] R. E. Cawagas, *On the structure and zero divisors of the Cayley-Dickson sedenion algebra*, Discuss. Math. Gen. Algebra Appl. **24** (2004) 251–265.

[Co] F. Cohen, *On the Whitehead square, Cayley-Dickson algebras, and rational functions*, Papers in honor of José Adem, Bol. Soc. Mat. Mexicana **37** (1992) 55–62.

[DDD] D. K. Biss, D. Dugger, and D. C. Isaksen, *Large annihilators in Cayley-Dickson algebras*, Comm. Algebra, to appear.

[DDDD] D. K. Biss, J. D. Christensen, D. Dugger, and D. C. Isaksen, *Large annihilators in Cayley-Dickson algebras II*, Bol. Soc. Mat. Mexicana, to appear.

[CD] K.-C. Chan and D. Ž. Đoković, *Conjugacy classes of subalgebras of the real sedenions*, Canad. Math. Bull. **49** (2006) 492–507.

[ES] P. Eakin and A. Sathaye, *On automorphisms and derivations of Cayley-Dickson algebras*, J. Algebra **129** (1990) 263–278.

[M1] G. Moreno, *The zero divisors of the Cayley-Dickson algebras over the real numbers*, Bol. Soc. Mat. Mexicana **4** (1998) 13–28.

[MG] G. Moreno and C. González-Flores, *Constructing zero divisors in the higher dimensional Cayley-Dickson algebras*, preprint, 2007.

Department of Mathematics, University of Chicago, Chicago, IL 60637, USA

Department of Mathematics, University of Western Ontario, London, Ontario N6A 5B7, Canada

Department of Mathematics, University of Oregon, Eugene, OR 97403, USA

Department of Mathematics, Wayne State University, Detroit, MI 48202, USA

E-mail address: `daniel@math.uchicago.edu`
E-mail address: `jdc@uwo.ca`
E-mail address: `ddugger@math.uoregon.edu`
E-mail address: `isaksen@math.wayne.edu`