GROTHENDIECK GROUPS OF COMPLEXES WITH NULL-HOMOTOPIES

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CONTENTS

1. INTRODUCTION

This paper is an addendum to [FH]. We reprove the main theorem of that paper by using Grothendieck groups of modules over certain DGAs. While the changes to the argument in [FH] are largely cosmetic, our approach shortens the proof from 25 pages down to 10 and greatly clarifies the overall structure. The authors of [FH] describe their proof as both "tedious" and "cumbersome", whereas the approach given here is neither of these.

1.1. **Statement of the main results.** Let R be a commutative ring, and let $S \subseteq R$ be a multiplicative system. Let $\mathcal{P}(R)$ denote the category of bounded chain complexes $P_\bullet$ such that $P_i = 0$ for $i < 0$ and where each P_i is a finitely-generated projective. Define the relative K-group $K(R \text{ on } S)$ as follows:

Definition 1.2. *Let $\mathcal{F}(R \text{ on } S)$ be the free abelian group on the isomorphism classes of objects $P_\bullet$ in $\mathcal{P}(R)$ having the property that $S^{-1}P_\bullet$ is exact. Let $K(R \text{ on } S)$ be the quotient of $\mathcal{F}(R \text{ on } S)$ by the following relations:*

(1) $[P_\bullet] = [P'_\bullet] + [P''_\bullet]$ whenever $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$ is a short exact sequence;
(2) $[P_\bullet] = 0$ if $H_*(P_\bullet) = 0$.

It is easy to see that if one alters the above definition to use complexes that are nonzero in negative dimensions (but are still bounded) then this gives rise to an isomorphic group. In this paper we will make the convention that all of our chain complexes vanish in negative dimensions, but this is only for simplicity of presentation.

For $n \geq 0$ define $\mathcal{P}(R, n)$ to be the full subcategory of $\mathcal{P}(R)$ consisting of complexes having $P_i = 0$ for $i \notin [0, n]$. Define $K(R \text{ on } S, n)$ in a similar way to Definition 1.2 but using this smaller collection of complexes. It is convenient to allow $n = \infty$ here, so that $K(R \text{ on } S) = K(R \text{ on } S, \infty)$. Notice that for $k \leq n \leq \infty$ the inclusion $\mathcal{P}(R, k) \hookrightarrow \mathcal{P}(R, n)$ induces a map $K(R \text{ on } S, k) \to K(R \text{ on } S, n)$.

Theorem 1.3 (Foxby-Halvorsen)**.** *The map $K(R \text{ on } S, k) \to K(R \text{ on } S, n)$ is an isomorphism for $1 \le k \le n \le \infty$.*

Foxby and Halvorsen actually prove a more general version of the above result that is necessary for their applications. Let $\underline{S} = (S_1, \ldots, S_d)$ be a d-tuple of multiplicative systems in R. Say that a chain complex $C_\bullet$ is **$\underline{S}$-exact** if $S_i^{-1} C_\bullet$ is exact for every i. Define $K(R \text{ on } \underline{S}, n)$ just as for $K(R \text{ on } S, n)$, except requiring the complexes $P_\bullet$ to all be $\underline{S}$-exact.

Theorem 1.4 (Foxby-Halvorsen)**.** *Let $\underline{S} = (S_1, \ldots, S_d)$ be a d-tuple of multiplicative systems. If $d \le k \le n \le \infty$ then $K(R \text{ on } \underline{S}, k) \to K(R \text{ on } \underline{S}, n)$ is an isomorphism.*

From the perspective of algebraic geometry we are looking at chain complexes of vector bundles on $\operatorname{Spec} R$ that are exact on a Zariski open subset. If every open subset had the form $\operatorname{Spec} S^{-1} R$ then one could be content with Theorem 1.3, but this is not the case. What is true instead is that every open subset of $\operatorname{Spec} R$ is a finite union of opens of the form $\operatorname{Spec} S^{-1} R$, and therein lies the importance of Theorem 1.4.

It seems worth pointing out that the lower bound on k in Theorem 1.4 is the best possible. Let F be a field and let $R = F[x_1, \ldots, x_d]_{(x_1, \ldots, x_d)}$. Let $S_i = \{x_i^r \mid r \ge 0\}$. The New Intersection Theorem says that all $\underline{S}$-exact complexes in $\mathcal{P}(R, d-1)$ are exact, and so $K(R \text{ on } S, d-1) = 0$. But a standard argument shows that $K(R \text{ on } S) \cong \mathbb{Z}$ (use the Resolution Theorem to reduce to the Grothendieck group of finite length R-modules, and then use dévissage).

Remark 1.5. The notation $K(R \text{ on } S)$ is used in [W] and was inspired by the notation for relative K-groups in [TT]. Because we will have various Grothendieck groups flying around in the rest of the paper, and because the choice between denoting these as "G" or "K" becomes awkward, we will just write everything with a "G". In particular, we use $G(R \text{ on } S, n)$ for the group denoted $K(R \text{ on } S, n)$ above.

1.6. **Outline of the argument.** For any $s \in R$ let T_s be the unique DGA whose underlying graded algebra is $R[e]$, where $\deg e = 1$, and whose differential satisfies $de = s$. Of course the differential is completely described by the formula

$$d(e^n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ se^{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

Note that for any $t \in R$ there is a map of DGAs $T_{st} \to T_s$ that sends e to te.

If C is a chain complex of R-modules, then giving a left T_s-module structure on C (that extends the R-module structure) is equivalent to specifying a null-homotopy for the multiplication-by-s map $C \to C$. This is just the statement that a map of R-DGAs $T_s \to \underline{\operatorname{Hom}}(C, C)$ is determined by the image of e, which must be an element $\tilde{e} \in \underline{\operatorname{Hom}}(C, C)_1$ such that $d\tilde{e} = s \cdot \operatorname{id}_C$. In particular, a T_1-structure on a chain complex C is the same as a contracting homotopy for C.

If S is a multiplicative system, let tS be the category whose objects are the elements of S and where the set of morphisms from s to t is $\{x \in S \mid sx = t\}$. This is the so-called *translation category* of the monoid S (under multiplication). Observe that tS is a filtered category, and that we have defined a functor $T \colon tS^{op} \to \mathcal{DGA}$ sending $s \mapsto T_s$.

Write $\mathcal{P}(T_s, n)$ for the category of dg-modules over T_s whose underlying chain complex lies in $\mathcal{P}(R, n)$. As always, we abbreviate $\mathcal{P}(T_s) = \mathcal{P}(T_s, \infty)$.

Definition 1.7. *Let $\mathcal{F}(T_s, n)$ be the free abelian group on isomorphism classes of objects in $\mathcal{P}(T_s, n)$. Define $G(T_s, n)$ to be the quotient of $\mathcal{F}(T_s, n)$ by the following relations:*

(1) $[X] = [X'] + [X'']$ whenever $0 \to X' \to X \to X'' \to 0$ is a short exact sequence in $\mathcal{P}(T_s, n)$;

(2) $[X] = 0$ if $H_*(X) = 0$.

The DGA maps $T_{st} \to T_s$ yield restriction functors $\mathcal{P}(T_s, n) \to \mathcal{P}(T_{st}, n)$. These are clearly exact, and so induce maps on Grothendieck groups
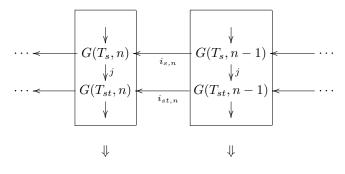
$$j_{st \leftarrow s} \colon G(T_s, n) \to G(T_{st}, n).$$

We will usually drop the subscripts and just write all such maps as "j". These maps assemble to define a functor $tS \to \mathcal{A}b$ given by $s \mapsto G(T_s, n)$.

A T_s-module is, in particular, a chain complex in $\mathcal{P}(R, n)$. The null-homotopy of the multiplication-by-s map shows that this complex becomes exact upon localization at S. So we have the evident homomorphism $G(T_s, n) \to G(R \text{ on } S, n)$. Obviously this gives a map $\operatorname{colim}_{tS} G(T_s, n) \to G(R \text{ on } S, n)$. A key step in proving the Foxby-Halvorsen results will be the following:

Proposition 1.8. *For any $1 \leq n \leq \infty$, the map $\operatorname{colim}_{tS} G(T_s, n) \to G(R \text{ on } S, n)$ is an isomorphism.*

Let $i_{s,n} \colon \mathcal{P}(T_s, n) \to \mathcal{P}(T_s, n+1)$ be the evident inclusion, and use the same notation for the induced map $G(T_s, n) \to G(T_s, n+1)$. Let $\mathbb{N}$ denote the category whose objects are the natural numbers $n \geq 1$ and where there is a unique map from n to $n+1$, for every n. The $i_{s,n}$ maps are clearly compatible with the change-of-rings maps induced by $T_{st} \to T_s$, and so putting them together we have a diagram $tS \times \mathbb{N} \to \mathcal{A}b$ sending $(s, n) \mapsto G(T_s, n)$. Here is a partial depiction of this large diagram:

$$
\begin{array}{ccccccc}
 & & \downarrow & & \downarrow & & \\
\cdots \longleftarrow & & G(T_s, n) \xleftarrow{\ i_{s,n}\ } & & G(T_s, n-1) & & \longleftarrow \cdots \\
 & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \\
\cdots \longleftarrow & & G(T_{st}, n) \xleftarrow{\ i_{st,n}\ } & & G(T_{st}, n-1) & & \longleftarrow \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & \Downarrow & & \Downarrow & & \\
\cdots \longleftarrow & & G(R \text{ on } S, n) \xleftarrow{\ i_n\ } & & G(R \text{ on } S, n-1) & & \longleftarrow \cdots
\end{array}
$$

The two framed boxes each depict a slice $tS \to \mathcal{A}b$ corresponding to fixing a value of n (note that because of typographical constraints we have not drawn a realistic model for tS—this category need not be linear). The colimit of these slices is written underneath them.

The second key component of our proof is the following:

Proposition 1.9. *For $n \geq 2$ there exist group homomorphisms $\mathcal{W}_n \colon G(T_s, n) \to G(T_{s^2}, n-1)$ such that $\mathcal{W}_n i = j$ and $i \mathcal{W}_n = j$.*

The theorem we are after is an immediate corollary of Propositions 1.8 and 1.9:

Corollary 1.10. *For any $n \geq 1$ the map $G(R \text{ on } S, n) \to G(R \text{ on } S, n+1)$ is an isomorphism. Consequently, the maps $G(R \text{ on } S, n) \to G(R \text{ on } S, \infty)$ are isomorphisms.*

Proof. It is trivial that $G(R \text{ on } S, \infty) = \operatorname{colim}_n G(R \text{ on } S, n)$, and so the second statement follows from the first. The proof of the first statement is formal, but we include it anyway. Let $\alpha \in G(R \text{ on } S, n+1)$. By Proposition 1.8 there exists an $s \in S$ and an $\alpha' \in G(T_s, n+1)$ that maps to α. Then $\mathcal{W}_{n+1}(\alpha')$ yields an element in $G(R \text{ on } S, n)$ that must map to α because $i \mathcal{W}_{n+1} = j$. This shows that i_n is surjective. Injectivity is similar: if $\beta \in G(R \text{ on } S, n)$ is in the kernel, then by Proposition 1.8 there exists $s \in S$ for which we can lift β to a $\beta' \in G(T_s, n)$ that is in the kernel of $i_{s,n}$. Then $0 = \mathcal{W}_{n+1}(0) = \mathcal{W}_{n+1} i_{s,n}(\beta') = j(\beta')$. Since β' and $j\beta'$ map to the same element in $G(R \text{ on } S, n)$, namely β, we must have $\beta = 0$. $\square$

We have now explained how the first main result (Theorem 1.3) is an immediate consequence of Propositions 1.8 and 1.9. The story is slightly trickier than we have indicated, however: a key insight is that Proposition 1.9 is actually *used to prove* Proposition 1.8. So in reality the structure of the argument is:

- Prove Proposition 1.9,
- Use Proposition 1.9 to prove Proposition 1.8, and finally
- Use Propositions 1.8 and 1.9 to deduce Theorem 1.3.

Everything we have said so far concerns the proof of Theorem 1.3, which is merely the $d = 1$ case of Theorem 1.4. However, the proof has been structured so that it adapts almost verbatim to the case of general d. Details are in Section 5.

Remark 1.11. At the risk of seeming repetitive we again point out that the basic idea for the proof is entirely due to [FH]. In particular, that paper gives a version of the maps $\mathcal{W}_n$. The contribution of the present paper is the repackaging of their construction into the above outline, which simplifies many technicalities.

2. Preliminary material

This section establishes some basic observations, constructions, and notation.

Proposition 2.1. *Let $P_\bullet$ be a bounded chain complex of finitely-generated projective R-modules such that $S^{-1} P_\bullet$ is exact. Then there exists $s \in S$ such that $P_\bullet$ has the structure of a T_s-module.*
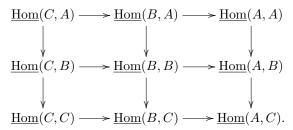
Proof. For R-modules M and N there is a natural map $S^{-1} \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$. This is clearly an isomorphism when $M = R$, and both the functors $S^{-1} \operatorname{Hom}_R(-, N)$ and $\operatorname{Hom}_{S^{-1}R}(S^{-1}(-), S^{-1}N)$ commute with finite direct sums and are left-exact. It follows that the natural map is an isomorphism for all finitely-presented modules—and in particular, for all finitely-generated projectives.

It follows from the preceding paragraph that the canonical map of chain complexes $S^{-1} \underline{\operatorname{Hom}}_R(P_\bullet, P_\bullet) \to \underline{\operatorname{Hom}}_{S^{-1}R}(S^{-1}P_\bullet, S^{-1}P_\bullet)$ is an isomoprhism. This uses that $P_\bullet$ is bounded and that its modules are finitely-generated. Since $S^{-1}P_\bullet$ is exact, it follows that $S^{-1} \underline{\operatorname{Hom}}_R(P_\bullet, P_\bullet)$ is exact. In particular, there exists an

element $e \in S^{-1} \underline{\mathrm{Hom}}(P_\bullet, P_\bullet)_1$ such that $de = \mathrm{id}$. The element e may be written as e'/s for some $e' \in \underline{\mathrm{Hom}}(P_\bullet, P_\bullet)_1$ and some $s \in S$, and we have $de' = s\,\mathrm{id}$ in $S^{-1} \underline{\mathrm{Hom}}(P_\bullet, P_\bullet)$. So there exists a $u \in S$ such that $u(de' - s\,\mathrm{id}) = 0$ in $\underline{\mathrm{Hom}}(P_\bullet, P_\bullet)$. This means $d(ue') = (us) \cdot \mathrm{id}$, and so ue' gives $P_\bullet$ the structure of a T_{us}-module. $\square$

Proposition 2.2. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of chain complexes of finitely-generated projective R-modules. If A is a T_s-module and C is a T_t-module, then there exists a T_{st}-module structure on B such that the exact sequence is an exact sequence of T_{st}-modules (where A and C become T_{st}-modules via restriction-of-scalars along the maps $T_{st} \to T_s$ and $T_{st} \to T_t$).*

Proof. Consider the following diagram of chain complexes:

$$
\begin{array}{ccccc}
\underline{\mathrm{Hom}}(C, A) & \longrightarrow & \underline{\mathrm{Hom}}(B, A) & \longrightarrow & \underline{\mathrm{Hom}}(A, A) \\
\downarrow & & \downarrow & & \downarrow \\
\underline{\mathrm{Hom}}(C, B) & \longrightarrow & \underline{\mathrm{Hom}}(B, B) & \longrightarrow & \underline{\mathrm{Hom}}(A, B) \\
\downarrow & & \downarrow & & \downarrow \\
\underline{\mathrm{Hom}}(C, C) & \longrightarrow & \underline{\mathrm{Hom}}(B, C) & \longrightarrow & \underline{\mathrm{Hom}}(A, C).
\end{array}
$$

Every row is a short exact sequence, as is every column. In the following argument we will just write (BC) as an abbreviation for $\mathrm{Hom}(B, C)$, and so forth. The argument is basically a diagram chase, but also making use of the fact that (BA) is a left-module over (AA).

We have elements $e \in (CC)_1$ such that $de = t \cdot \mathrm{id}$ and $f \in (AA)_1$ such that $df = s \cdot \mathrm{id}$. Our goal is to produce an element $g \in (BB)_1$ such that $dg = st \cdot \mathrm{id}_B$ and such that (1) both g and se map to the same element of $(BC)_1$, and (2) both g and tf map to the same element of $(AB)_1$. This will prove the proposition.

Let $e' \in (CB)_1$ be a preimage for e, and let e'' be the image of e' in $(BB)_1$. Write $a = de''$. One sees readily that both a and $t \cdot \mathrm{id}_B$ map to the same element of $(BC)_0$, and so $a - t \cdot \mathrm{id}_B$ is the image of an element $b \in (BA)_0$. This b must be a cycle, since $a - t \cdot \mathrm{id}_B$ is.

Next, consider the element $f \cdot b \in (BA)_1$. This satisfies $d(fb) = sb$ by the Leibniz rule. If x is the image of fb in $(BB)_1$ then $dx = sa - st\,\mathrm{id}_B$, and so $d(se'' - x) = st \cdot \mathrm{id}_B$. Let $z = se'' - x$. The last thing to do is to calculate the image of z in (AB); this is the same as the image of $-x$, which by commutativity of the upper left square is the same as the image of $-f \cdot b$ in (AB).

Let b' be the image of b in (AA). Both b' and $-t \cdot \mathrm{id}_A$ map to $-t \cdot i$ under $(AA) \to (AB)$, where i is $A \hookrightarrow B$. So $b' = -t \cdot \mathrm{id}_A$ by injectivity. The image of $f \cdot b$ in (AA) is then $f \cdot b' = -tf$. So z and tf map to the same element in (AB). $\square$

Remark 2.3. A common use of Proposition 2.2 occurs when X is a T_s-module, Y is a T_t-module, and $f \colon X \to Y$ is a map of chain complexes over R. Let Cf be the mapping cone of f, which sits in a short exact sequence $0 \to Y \to Cf \to \Sigma X \to 0$. The above proposition guarantees us a T_{st}-module structure on Cf so that this is an exact sequence of T_{st}-modules. In this case it is useful to be completely explicit about what the module structure is. To this end, write $(Cf)_n = Y_n \oplus \sigma X_{n-1}$, where the σ is simply a marker that will be useful in formulas. For example, the differential on Cf is given by $d_{Cf}(y) = dy$ for $y \in Y_n$ and $d_{Cf}(\sigma x) = f(x) - \sigma dx$ for $x \in X_{n-1}$. The σ-notation is also used in $(\Sigma X)_n = \sigma X_{n-1}$, where the differential

is $d(\sigma x) = -\sigma(dx)$. Likewise, the T_s-module structure on ΣX is $e.(\sigma x) = -\sigma(e.x)$. If one regards the symbol σ as having degree 1 and satisfying the usual Koszul sign conventions, all of the above formulas are easy to remember.

If E denotes the generator for T_{st}, define a module structure on Cf by

$$\begin{cases} E.y = s.ey \text{ for } y \in Y_n, \text{ and} \\ E.(\sigma x) = ef(ex) - t.\sigma(ex) \text{ for } x \in X_{n-1}. \end{cases}$$

One readily checks that this is indeed a T_{st}-module structure, and that it has the desired properties.

In the case where $s = t$ and f is a map of T_s-modules we can do slightly better. Here one can put a T_s-module structure on Cf via the formulas $E.y = ey$ and $E.(\sigma x) = -\sigma(ex)$. In other words, we do not need to pass to T_{s^2} here.

Remark 2.4. Here is an observation that will be used often. If X is an object in $\mathcal{P}(T_s, n-1)$ then the cone on the identity map sits in a short exact sequence $0 \to X \to C(\mathrm{id}) \to \Sigma X \to 0$. This is an exact sequence of T_s-modules using the last paragraph of Remark 2.3, and all the modules lie in $\mathcal{P}(T_s, n)$. We obtain $[\Sigma X] = -[X]$ in $G(T_s, n)$.

Finally, consider an X in $\mathcal{P}(T_s, n)$. The subcomplex $0 \to X_n \xrightarrow{d} d(X_n) \to 0$ of X (concentrated in degrees n and $n-1$) is a T_s-submodule, and therefore the quotient of X by this submodule has an induced T_s-structure. This quotient is the chain complex $\tau_{\leq n-1} X$, which coincides with X in degrees less than $n-1$ and equals $X_{n-1}/d(X_n)$ in degree $n-1$. The complex $\tau_{\leq n-1} X$ need not be in $P(T_s, n-1)$, as there is no guarantee that $X_{n-1}/d(X_n)$ is a projective R-module.

However, assume that X is contractible as a chain complex. Then X admits a structure of T_1-module (not related to the T_s-structure), and this consists of a collection of maps $X_{i-1} \to X_i$ corresponding to left-multiplication by e. If $x \in X_n$ then $ex = 0$ and so $0 = d(ex) = x - e(dx)$, hence $e.dx = x$. So the map $X_{n-1} \to X_n$ is a left inverse for $d \colon X_n \to X_{n-1}$, and hence $X_{n-1}/d(X_n)$ is projective.

For P an R-projective let $D_s^n(P)$ be the complex $0 \to P \xrightarrow{\mathrm{id}} P \to 0$, concentrated in degrees n and $n-1$, with T_s-module structure given by the multiplication-by-s map. We have proven the following:

Proposition 2.5. *Let $X \in \mathcal{P}(T_s, n)$ and assume the underlying chain complex of X is contractible. Then there is a short exact sequence in $\mathcal{P}(T_s, n)$ of the form*

$$0 \to D_s^n(X_n) \xrightarrow{f} X \longrightarrow \tau_{\leq n-1} X \to 0$$

where f equals the identity in degree n and equals d in degree $n-1$.

One may continue with the above process, producing a similar exact sequence with $\tau_{\leq n-1} X$ as the middle term and then proceeding inductively. So every contractible T_s-module may be built up via extensions from the complexes $D_s^k(P)$. This proves the following:

Corollary 2.6. *If Definition 1.7 is changed so that relation (2) is only imposed for modules of the form $D_s^k(P)$, $1 \leq k \leq n$ and P a finitely-generated R-projective, then this yields the same group $G(T_s, n)$.*

Remark 2.7 (Duality). If X is a T_s-module then $\mathrm{Hom}(X, R)$ has an induced T_s-structure: one simply turns all the arrows around. This gives a duality functor on

each of the categories $\mathcal{P}(T_s, n)$. It is useful to remember that every construction in this paper has a dual version. For example, this applies to Proposition 2.5 above.

3. The construction of the $\mathcal{W}$ maps

In this section we will construct the maps $\mathcal{W}\colon G(T_s, n) \to G(T_{s^2}, n-1)$ for $n \geq 2$ and prove Proposition 1.9.

For $n \geq 2$ we describe a functor $\Gamma\colon \mathcal{P}(T_s, n) \to \mathcal{P}(T_{s^2}, n-1)$. For X in $\mathcal{P}(T_s, n)$ this functor takes the top degree X_n and "folds" it down into degree $n-2$, interchanging the d and e operators on this piece. This doesn't quite yield a T_s-module, but a little fiddling yields a T_{s^2}-structure. For $s = 1$ this construction appears in [D, 2.10], and a related construct is in [A, proof of Lemma 2.6.9].

To avoid some confusion let us write e for the polynomial generator of T_s and E for the corresponding generator of T_{s^2}. If $X \in \mathcal{P}(T_s, n)$ define

$$(\Gamma X)_i = \begin{cases} X_i & \text{if } 0 \leq i \leq n-3 \text{ or } i = n-1, \\ X_{n-2} \oplus X_n & \text{if } i = n-2, \\ 0 & \text{otherwise.} \end{cases}$$

The differential $(\Gamma X)_i \to (\Gamma X)_{i-1}$ coincides with the one on X (denoted d_X) for $i \leq n-3$. For $(a, b) \in X_{n-2} \oplus X_n = (\Gamma X)_{n-2}$ we set $d(a, b) = d_X a$, and for $a \in X_{n-1} = (\Gamma X)_{n-1}$ we set $da = (d_X a, ea)$. This clearly makes ΓX into a chain complex of finitely-generated, projective R-modules. To complete the construction we need to describe the T_{s^2}-module structure. For $a \in (\Gamma X)_i$ set

$$E.a = \begin{cases} s(e.a) & \text{if } 0 \leq i \leq n-3, \\ s(e.a) - e^2.da & \text{if } a \in X_{n-2} \subseteq (\Gamma X)_{n-2}, \\ s.da & \text{if } a \in X_n \subseteq (\Gamma X)_{n-2} \\ 0 & \text{otherwise.} \end{cases}$$

It is routine, although slightly tedious, to check that this really is the structure of a T_{s^2}-module (for $u \in X_{n-2}$ use that $0 = d(e^3 u) = se^2 u - e^3 du$). The fact that Γ is a functor is then self-evident. Moreover, it is clearly an exact functor: if $0 \to X' \to X \to X'' \to 0$ is a short exact sequence of T_s-modules then $0 \to \Gamma X' \to \Gamma X \to \Gamma X'' \to 0$ is still short exact. Finally, we note that if $i\colon \mathcal{P}(T_s, n-1) \to \mathcal{P}(T_s, n)$ is the evident inclusion then the composition

$$\mathcal{P}(T_s, n-1) \xrightarrow{i} \mathcal{P}(T_s, n) \xrightarrow{\Gamma} \mathcal{P}(T_{s^2}, n)$$

is just the restriction-of-scalars functor j.

Write $K(s)$ for the Koszul complex $R \xrightarrow{s} R$, concentrated in degrees 0 and 1. It has an evident structure of T_s-module, since $K(s)$ is just $T_s/(e^2)$. Let us record the following simple calculation:

Lemma 3.1. *If P is an R-module then $\Gamma(D_s^n P) \cong \Sigma^{n-2}(K(s) \otimes_R P)$ (as T_{s^2}-modules), for $n \geq 2$.*

Remark 3.2 (A better approach to ΓX)**.** The definition of Γ seems to have come out of thin air. To understand it better, notice that for any $X \in \mathcal{P}(T_s, n)$ there is a natural chain map $f\colon X \to \Sigma^{n-1}(K(s) \otimes_R X_n)$. It equals the identity in degree n and is given (up to sign) by left-multiplication by e in degree $n-1$. Note that

this is not a map of T_s-modules. However, let Cf be the mapping cone of f. By Remark 2.3 there is a canonical structure of T_{s^2}-module on Cf such that

$$(3.3) \qquad 0 \to \Sigma^{n-1}(K(s) \otimes_R X_n) \to Cf \to \Sigma X \to 0$$

is an exact sequence of T_{s^2}-modules.

Note that $(Cf)_{n+1} = X_n$, and consider the canonical map of T_{s^2}-modules $D_{s^2}^{n+1}(X_n) \to Cf$. This is an inclusion (because f was the identity in degree n) and the image is a T_{s^2}-subcomplex, so we may consider the quotient module. An easy calculation shows that this quotient is isomorphic to $\Sigma(\Gamma X)$:

$$(3.4) \qquad 0 \to D_{s^2}^{n+1}(X_n) \to Cf \to \Sigma(\Gamma X) \to 0.$$

Each of the complexes in both (3.3) and (3.4) are concentrated in degrees 1 through $n+1$; so if we desuspend everything we may regard these as exact sequences in $\mathcal{P}(T_{s^2}, n)$. Since $D_{s^2}^{n+1}(X_n)$ is contractible we have $[\Sigma^{-1}Cf] = [\Gamma X]$ in $G(T_{s^2}, n)$, and (3.3) then gives the identity

$$(3.5) \quad [X] = [\Sigma^{-1}Cf] - [\Sigma^{n-2}K(s) \otimes_R X_n] = [\Gamma X] + (-1)^{n-1}[K(s) \otimes_R X_n]$$

in $G(T_{s^2}, n)$.

We are ready to construct the $\mathcal{W}$ maps and prove they have the desired properties:

Proof of Proposition 1.9. For every $X \in \mathcal{P}(T_s, n)$ let

$$\mathcal{W}_n(X) = [\Gamma X] - [\Sigma^{n-2}K(s) \otimes_R X_n] \in G(T_{s^2}, n-1).$$

This assignment is additive since both terms in the definition of $\mathcal{W}_n(X)$ are additive. Using Corollary 2.6, the assignment will descend to the Grothendieck group if we verify $\mathcal{W}_n(X) = 0$ for $X = D_s^k(P)$, where $1 \le k \le n$ and P is a finitely-generated projective. If $k < n$ this is trivial because then $\Gamma X = X$ (which is contractible) and $X_n = 0$. For $k = n$ the claim follows from Lemma 3.1.

So $\mathcal{W}_n$ induces a group homomorphism $G(T_s, n) \to G(T_{s^2}, n-1)$, which we will also call $\mathcal{W}_n$. The identity $\mathcal{W}_n i = j$ is an immediate consequence of $\Gamma i = j$, and $i\mathcal{W}_n = j$ is simply (3.5). $\square$

4. Proof of the colimit result

In this section we prove Proposition 1.8.

Lemma 4.1. *Let $P_\bullet \in \mathcal{P}(T_s, n)$, and suppose that there is another T_s-module structure on the same underlying chain complex of R-modules. Denote the two T_s-modules as $P_\bullet^1$ and $P_\bullet^2$. Then the images of $[P_\bullet^1]$ and $[P_\bullet^2]$ in $\mathrm{colim}_s G(T_s, n)$ are the same.*

Proof. Start by considering the short exact sequence $0 \to P_\bullet \to C(\mathrm{id}) \to \Sigma P_\bullet \to 0$ where the middle term is the cone on the identity map. The second two complexes in the sequence have length $n + 1$. Equip the left term $P_\bullet$ with the first T_s-module structure, and equip the right term $\Sigma P_\bullet$ with the second T_s-module structure (corresponding to $\Sigma P_\bullet^2$). By Proposition 2.2 there is a T_{s^2}-module stucture on $C(\mathrm{id})$ such that the above is an exact sequence of T_{s^2}-modules. Since $C(\mathrm{id})$ is contractible we therefore have

$$[P_\bullet^1] = -[\Sigma P_\bullet^2] = [P_\bullet^2]$$

in $G(T_{s^2}, n+1)$. It is perhaps better to write this as $j_{s^2 \leftarrow s} i_s[P^1_\bullet] = j_{s^2 \leftarrow s} i_s[P^2_\bullet]$, where i_s is the map $G(T_s, n) \to G(T_s, n+1)$. By commuting the j and i we then get $i_{s^2} j_{s^2 \leftarrow s}[P^1_\bullet] = i_{s^2} j_{s^2 \leftarrow s}[P^2_\bullet]$.

Now apply the map $\mathcal{W}_{s^2} \colon G(T_{s^2}, n+1) \to G(T_{s^4}, n)$ to get

$$\mathcal{W}_{s^2} i_{s^2} j_{s^2 \leftarrow s}[P^1_\bullet] = \mathcal{W}_{s^2} i_{s^2} j_{s^2 \leftarrow s}[P^2_\bullet].$$

By Proposition 1.9 one has $\mathcal{W}_{s^2} i_{s^2} = j_{s^4 \leftarrow s^2}$, and so the above identity simply says $j_{s^4 \leftarrow s}[P^1_\bullet] = j_{s^4 \leftarrow s}[P^2_\bullet]$. Consequently, $[P^1_\bullet] = [P^2_\bullet]$ in $\mathrm{colim}_s G(T_s, n)$. $\qquad \square$

Proof of Proposition 1.8. We must prove that the map $j \colon \mathrm{colim}_s G(T_s, n) \to G(R \text{ on } S, n)$ is an isomorphism, and we will do this by constructing an inverse. Let $P_\bullet$ be a bounded complex of finitely-generated projectives that is S-exact. Then by Proposition 2.1 $P_\bullet$ may be given the structure of T_s-module, for some $s \in S$. By Lemma 4.1 the class of this T_s-module in $\mathrm{colim}_s G(T_s, n)$ is independent of the choice of T_s-module structure: write this class as $\{P\}$.

We must show that the assignment $P_\bullet \mapsto \{P\}$ is additive and sends contractible complexes to zero. The latter is trivial. For the former, let $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$ be an exact sequence in $\mathcal{P}(R \text{ on } S, n)$. Choose a T_s-module structure on $P'_\bullet$ and on $P''_\bullet$. By Proposition 2.2 there is a T_{s^2}-module structure on $P_\bullet$ making the above an exact sequence of T_{s^2}-modules. So we have $[P_\bullet] = [P'_\bullet] + [P''_\bullet]$ in $G(T_{s^2}, n)$, and this immediately yields $\{P\} = \{P'\} + \{P''\}$. So we have constructed a map $\lambda \colon G(R \text{ on } S, n) \to \mathrm{colim}_s G(T_s, n)$. The equation $j\lambda = \mathrm{id}$ is obvious, and the equation $\lambda j = \mathrm{id}$ is immediate from the fact that the definition of λ did not depend on the choice of T_s-module structure. $\qquad \square$

5. PROOF OF THE GENERAL FOXBY-HALVORSEN RESULT

Fix $d \geq 1$ and let $\underline{s} \in R^n$. Define $T_{\underline{s}}$ to be the free DGA over R generated by elements $e_1, \ldots, e_d$ of degree 1 satisfying $d(e_i) = s_i$. As an algebra $T_{\underline{s}}$ is the free (non-commutative) R-algebra on d variables. Note that to give a $T_{\underline{s}}$-module structure on a chain complex $C_\bullet$ is the same as giving null-homotopies for the multiplication-by-s_i maps with $1 \leq i \leq d$. Because the null-homotopies are independent of each other, the analogs of the results in Section 2 are all straightforward consequences of what we have already proven.

If $\underline{s}, \underline{t} \in R^n$ write $\underline{st}$ for the tuple whose ith element is $s_i \cdot t_i$. There are canonical DGA maps $T_{\underline{st}} \to T_{\underline{s}}$ sending $e_i \mapsto t_i e_i$. If $S_1, \ldots, S_d$ are multiplicative systems then these maps assemble to give a diagram of DGAs $(tS_1 \times \cdots \times tS_d)^{op} \to \mathcal{DGA}$. Note that $tS_1 \times \cdots \times tS_d$ is a product of filtered categories, hence filtered.

Let $\mathcal{P}(T_{\underline{s}}, n)$ and $G(T_{\underline{s}}, n)$ be defined in the evident way, generalizing our notation from Section 1.6. We obtain a diagram of abelian groups $tS_1 \times \cdots \times tS_d \times \mathbb{N} \to \mathcal{A}b$, with $(s_1, \ldots, s_d, n) \mapsto G(T_{\underline{s}}, n)$. We again use j for any map in this diagram between objects with n fixed, and i for any map between objects with $\underline{s}$ fixed.

Theorem 5.1.

(a) For $n \geq d+1$ there are maps $\mathcal{W} \colon G(T_{\underline{s}}, n) \to G(T_{\underline{s}^2}, n-1)$ satisfying $\mathcal{W} i = j$ and $i \mathcal{W} = j$.

(b) For $n \geq d$ the maps $\mathrm{colim}_{\underline{s}} G(T_{\underline{s}}, n) \to G(R \text{ on } \underline{S}, n)$ are isomorphisms.

(c) The maps $G(R \text{ on } \underline{S}, n) \to G(R \text{ on } \underline{S}, n+1)$ are isomorphisms, for all $n \geq d$.

The proof of the above result closely parallels what we did for $d = 1$. The main component is the construction of a functor $\Gamma \colon P(T_{\underline{s}}, n) \to P(T_{\underline{s}^2}, n-1)$ that "folds"

the degree n piece down into the lower degrees. The only difference from $d = 1$ is that this functor is a bit more complicated. To see why, note that the maps $e_1, \ldots, e_d \colon X_{n-1} \to X_n$ force us to now fold d different copies of X_n into degree $n - 2$. The e_i maps on these folded terms will all be the old differential d_X, but one quickly finds that this does not give anything close to a $T_{\underline{s}^2}$-structure. It turns out that in order to fix this one needs to put certain "interaction" terms into the lower degrees—these interaction terms end up forming a higher-dimensional Koszul complex (see Remark 5.6 for a picture).

We start by establishing some notation. For $\underline{s} \in R^n$ let $K(\underline{s})$ denote the Koszul complex for $s_1, \ldots, s_d$. This is just the quotient of $T_{\underline{s}}$ by the relations $e_i e_j + e_j e_i = 0$ and $e_i^2 = 0$, $1 \le i, j \le d$. In particular, note that $K(\underline{s})$ becomes a $T_{\underline{s}}$-module in the evident way.

Lemma 5.2. *Let X be a $T_{\underline{s}}$-module. There there is a chain map $K(\underline{s}) \otimes_R X_0 \to X$ that is the identity in degree 0, and is natural in X. (Warning: This is not a map of $T_{\underline{s}}$-modules).*

Proof. For $1 \le i_1 < \cdots < i_k \le d$ and $u \in X_0$, send $e_{i_1} \cdots e_{i_k} \otimes u \mapsto e_{i_1} \cdots e_{i_k}.u$. The Leibniz rule and the fact that $du = 0$ readily show this to be a chain map. $\square$

What we actually need is the dual of the above lemma: if $X \in \mathcal{P}(T_s, n)$ then there is a natural chain map $f \colon X \to \Sigma^{n-d}(\Omega(\underline{s}) \otimes_R X_n)$ where $\Omega(\underline{s})$ is the dual of the Koszul complex. (As it happens, the Koszul complex is self-dual—but we do not need this fact.) This dual result follows immediately, using Remark 2.7. But in the interest of beng concrete let us describe the chain map f more precisely. This description is unnecessary for our application, but it seems worth including.

Let $e_1, \ldots, e_d$ be the standard basis for R^d, so that $\underline{s} = \sum_j s_j e_j$. Define the complex $\Omega(\underline{s})$ by $\Omega(\underline{s})_i = \wedge^{n-i} R^d$, where the differential $d \colon \Omega(\underline{s})_i \to \Omega(\underline{s})_{i-1}$ sends $\omega \mapsto \omega \wedge \underline{s}$. The complex $\Omega(\underline{s})$ is the dual of $K(\underline{s})$, but in fact we also have $K(\underline{s}) \cong \Omega(\underline{s})$. The isomorphism is via the Hodge $*$-operator: if $\omega = e_{i_1} \wedge \cdots \wedge e_{i_k}$ define $*\omega = e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}$ to be the unique wedge product of this form having the property that $\omega \wedge *\omega = (-1)^{k(n-k)} e_1 \wedge \cdots \wedge e_n$. One readily checks that the assignment $\omega \mapsto *\omega$ is a chain map $K(\underline{s}) \to \Omega(\underline{s})$, and then it is clearly an isomorphism. We use this observation only to see that $\Omega(\underline{s})$ has a $T_{\underline{s}}$-structure. [The structure has $e_r \cdot (e_{j_1} \wedge \cdots \wedge e_{j_t})$ equal to zero if e_r does not appear in the wedge product, and equal to $(-1)^{a+t} e_{j_1} \wedge \cdots \widehat{e_{j_a}} \wedge \cdots \wedge e_{j_t}$ if $j_a = r$.]

Lemma 5.3. *Let X be a $T_{\underline{s}}$-module such that $X_i = 0$ for $i > n$. Then there is a chain map $X \to \Sigma^{n-d}(X_n \otimes \Omega(\underline{s}))$ that equals the identity in degree n, and this map is natural in X. (Warning: this is not a map of $T_{\underline{s}}$-modules).*

Proof. For any $u \in X_{n-k}$ and any $j_1, \ldots, j_{k+1} \in [1, d]$ the Leibniz rule yields

$$(5.4) \quad 0 = d(0) = d(e_{j_1} \cdots e_{j_{k+1}}.u)$$

$$= \sum_{a=1}^{k+1} (-1)^{a-1} s_{j_a} \cdot e_{j_1} \cdots \widehat{e_{j_a}} \cdots e_{j_{k+1}}.u + (-1)^{k+1} e_{j_1} \cdots e_{j_{k+1}}.du.$$

Define $f \colon X_{n-k} \to X_n \otimes \Omega(\underline{s})_k$ by the formula

$$u \mapsto \sum_{i_1 < \cdots < i_k} (e_{i_1} e_{i_2} \cdots e_{i_k}.u) \otimes (e_{i_1} \wedge \cdots \wedge e_{i_k}).$$

The verification that this is a chain map is routine, using (5.4). $\square$

Proof of Theorem 5.1. Once (a) is established, the proofs of (b) and (c) are the same as the $d = 1$ case we have already done. So the only work is in proving (a).

Let X be an object in $\mathcal{P}(T_{\underline{s}}, n)$, and let $f \colon X \to \Sigma^{n-d}(X_n \otimes \Omega(\underline{s}))$ be the map provided by Lemma 5.3. Let Cf be the mapping cone of f, which comes equipped with a canonical $T_{\underline{s}^2}$-module structure. The fact that f equals the identity in the top degree yields that Cf has the complex $D_{\underline{s}^2}^{n+1}(X_n)$ as a subcomplex. Indeed, one readily checks that this is a sub-$T_{\underline{s}^2}$-module. Let QX denote the quotient, so that we have two short exact sequences of $T_{\underline{s}^2}$-modules

(5.5) $$0 \to \Sigma^{n-d}(X_n \otimes \Omega(\underline{s})) \to Cf \to \Sigma X \to 0, \quad \text{and}$$
$$0 \to D_{\underline{s}^2}^{n+1}(X_n) \to Cf \to QX \to 0.$$

Note that QX is concentrated in degrees 1 through n (this uses that $n \geq d + 1$), and let ΓX be the desuspension of QX.

It is obvious that ΓX is functorial in X, and it is also obvious that $\Gamma(-)$ is additive (one only has to note that in each degree ΓX is a direct sum of certain homogeneous components of X). Define

$$\mathcal{W}_n(X) = [\Gamma X] + [\Sigma^{n-d}(X_n \otimes \Omega(\underline{s}))] \in G(T_{\underline{s}^2}, n).$$

The proof that this descends to a map $G(T_{\underline{s}}, n) \to G(T_{\underline{s}}, n-1)$ is exactly the same as the $d = 1$ case—it boils down to the very simple computation that $\Gamma(D_{\underline{s}}^n P) \cong \Sigma^{n-d}(K(\underline{s}) \otimes_R P)$ as $T_{\underline{s}^2}$-modules. It is evident that $\mathcal{W}i = j$. The identity $i\mathcal{W} = j$ follows immediately from the two short exact sequences in (5.5) once one realizes that the complexes all lie in degrees 1 through $n + 1$, and therefore the sequences may be desuspended to give exact sequences in $\mathcal{P}(T_{\underline{s}}^2, n)$. This proves (a). $\square$

Remark 5.6. The complex ΓX looks as shown below, where the differential has both the horizontal components and the cross-term depicted diagonally:

$$X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_{n-d-1} \leftarrow \cdots \leftarrow X_{n-3} \leftarrow X_{n-2} \leftarrow X_{n-1}$$

$$\oplus \qquad\qquad \oplus \qquad\qquad \oplus$$

$$X_n \otimes \wedge^d R^d \leftarrow \cdots \leftarrow X_n \otimes \wedge^2 R^d \leftarrow X_n \otimes \wedge^1 R^d$$

We are finally ready to complete our proof of the main result:

Proof of Theorem 1.4. This is a formal consequence of Theorem 5.1; see the proof of Corollary 1.10. $\square$

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