All solutions should be provided in Sage. Remember to answer all the questions posed in each problem.

1. Find all digits (0–9) for $a$ and $b$ such that the decimal number $a43b16$ is divisible by 21.

2. For what positive integers $n$ does the decimal expansion for $\frac{1}{n}$ end in all zeros? Use Sage to explore this. I suggest making a chart of the decimal expansions for $n \leq 100$, and then look for patterns.

3. (a) If $a$ and $b$ are randomly chosen from the interval $[-1, 1]$, what is the probability that the point $(a, b)$ lies inside the disk $x^2 + y^2 \leq 1$? Draw a picture and answer this using basic geometry.

   (b) Write a Sage program that approximates $\pi$ by actually performing the above experiment. Your code should do the following:
   
   1. Initialize a graphics object, and start a count at zero.
   2. Randomly choose $a$ and $b$ in $[-1, 1]$ (this part might take a little ingenuity, since you only have the functions `randint` and `random`).
   3. If the point $(a, b)$ lies inside the unit disk, increase the count by 1 and plot the point in red. If it doesn’t lie inside the disk, plot it in blue (but don’t increase the count).
   4. Repeat the above steps $n$ times, and compute the approximate probability $\frac{\text{count}}{n}$. Then use this to calculate an approximation for $\pi$, and print this approximation.

4. In a certain small country, license plates look like $ABC789$ where the first three spots are letters $A – F$ and the last three spots are digits 0–9. Let’s call the string formed in the first half the “alphabetical part”, and the number formed in the second half the “numerical part.”

   (a) What is the probability that a randomly chosen license plate has the sum of the digits at least 15? Use Sage to run a simulation that approximates this. (The answer should be around 0.4).

   (b) What is the probability that a randomly chosen license plate has at least one vowel? Again, use Sage to run a simulation. (The answer should be more than 0.5).

5. An integer $n$ is a “sum of two squares” if it can be written as $n = a^2 + b^2$ for some integers $a$ and $b$. So 5 = $1^2 + 2^2$ and 9 = $0^2 + 3^2$ are both sums of two squares, but 3 and 6 are not.

   (a) Write a Sage function `sum_squares` that takes an integer $n$ and returns True or False depending on whether $n$ is a sum of two squares or not.

   (b) How many numbers from 1 to 100 are sums of two squares? How many numbers from 1 to 1000 are sums of two squares?
(c) Repeat part (a) for sums of three squares. Also, what is the first number above 100 that is not the sum of three squares?

(d) Repeat part (a) for sums of four squares.

[Note: Depending on how you answer part (a), counting sums of squares under 1000 could take a few minutes of computing time (but it shouldn’t take more than 5 minutes at the most). So don’t be overly impatient.]