The first part of this worksheet just recalls what we worked on last week in class. Jump to the next page for the assignment.

A graph is a collection of vertices and edges. Each edge has two ends, which are vertices. Here is an example:

A vertex labelling of a graph is a labelling that assigns a real number to each vertex. An edge labelling of a graph is similar, but it assigns a real number to each edge. Here are examples of each:

If you start with a vertex labelling of a graph, you can get an associated edge labelling by the following procedure: label each edge $e$ by the sum of the labels on the two vertices that form the ends of $e$. Here is the beginning of an example:

Complete the example by appropriately labelling the other edges in the graph on the right.

Here is a question we will explore: Is this process reversible? That is to say, does every edge labelling come from a vertex labelling in this way?

1. Explore the question when the graph is the triangle, with three vertices and three edges. Start with a triangle where the edges are labelled with 3, 5, and 2 and try to find a corresponding vertex labelling. Some questions to think about:
(a) Is there an efficient way to determine whether an appropriate vertex labelling exists, and what it is?

(b) What happens when you change the labels 3, 5, and 2? Can you give an algorithm for solving the problem no matter what the labels are?

2. What happens when the graph is a square?

3. Think about other graphs. Which ones have the property that every edge labelling comes from a vertex labelling? Which ones have the property that some edge labellings do not come from a vertex labelling, and some do? In each case, can you completely characterize all such graphs?

The above questions represent what we did in class last week, though we didn’t quite finish answering Question #3. Let’s say that a graph is solvable if every edge labelling comes from a vertex labelling, in the way we have been talking about. So we know that a triangle is solvable, and a square is not solvable. We also talked about the fact that trees are solvable, and the $n$-cycle is solvable precisely when $n$ is odd. For your homework, work on the following problems:

4. Consider the dumbell graph

Determine whether or not this is solvable. If it is solvable, explain why. If it is not solvable, give a specific edge labelling that does not come from a vertex labelling.

5. Repeat the above question for the following graph:

6. Try to make some conjectures about general classes of graphs that are solvable, or classes that aren’t solvable.
7. Finally, consider **directed graphs**. A directed graph is a graph together with an arrow that is marked on each edge, pointing in one direction. Given a vertex labelling, now we will give a different procedure for making an edge labelling: to an edge $e$ pointing from $a$ to $b$, we assign the difference

“label of $b$” $-$ “label of $a$”.

Here is an example:

Again, explore the question of whether every edge labelling comes from a vertex labelling. Try it for triangles, squares, and so forth, and try to come up with a general characterizations of the graphs where this always works.