MATH 206
Monday, October 8
Week 3 Worksheet

1. Practice some modular arithmetic if you haven’t learned it before. Make sure you can figure out \(3 +_7 6\), \(3 -_7 6\), \(2 \cdot _8 7\), and similar things. Let \(\mathbb{Z}_n = \{0, 1, 2, \ldots, n - 1\}\). Solve \(6 \cdot _7 x = 1\) for \(x \in \mathbb{Z}_7\).

2. You have a pyramid of six numbers: three on the bottom level, two in the middle level, and one on top. The rule of the pyramid is that every number is the sum of the two just below it in \(\mathbb{Z}_6\). Can you make a pyramid like this using the numbers 0, 1, 2, 3, 4, 5, where each number is used exactly once?

3. Fix a positive integer \(n \geq 3\). Two players alternate naming a number from the set \(\{1, 2, \ldots, n - 1\}\) and after each move the running total is computed modulo \(n\). You can’t play a number that has already been played. The first player to arrive at a running total of 0 loses.

   Question: What kind of outcomes can the game have? Does either of the two players have a winning strategy? Does the answer depend on \(n\)?

4. What is the remainder of \(100 \cdot 101 \cdot 102 \cdot 103 \cdot 104\) when divided by 99? What is the remainder when \(3^{100}\) is divided by 7?

5. In a non-leap year, will there always be a Friday the 13th? How many Friday the 13ths could there be? How do the answers change (if at all) during a leap year?

   [Hint: This problem takes a little legwork, but one approach is to use arithmetic in \(\mathbb{Z}_7\). Encode the days of the week as Sunday=0, Monday=1, and so forth. Let \(A\) denote the number corresponding to the day on which January 13th falls. What next? ]

   Bonus question: What is the greatest number of months that can go by without having a Friday the 13th?

6. On a planet far, far away the only inhabitants are chameleons. They come in three colors: green, yellow, and red. On this planet, there is a law which governs how chameleons can change their color: whenever two chameleons of different colors meet, they must both change to the third color. Given that the planet currently has just 4 green, 5 yellow, and 5 red chameleons, is it possible that after some period of time that ALL of the chameleons on the planet will have the same color?

   Do the same if the initial colors were 4 green, 5 yellow, and 6 red.

   Explain and generalize!

7. I have a machine that takes as input a card on which is written an ordered pair \((x, y)\) of positive integers. The machine can do three different operations:
Operation 1: The machine adds 1 to both numbers, outputting a new card with the numbers $(x + 1, y + 1)$ on it. You get to keep the original card too for later use if you need it.

Operation 2: If both the numbers $x$ and $y$ are even, the machine halves them both, outputting a new card with $(x/2, y/2)$ on it. You again get to keep the original card.

Operation 3: You can input two cards $(x, y)$ and $(y, z)$ into the machine such that the second coordinate of the first matches the first component of the second. Then it outputs the new card with numbers $(x, z)$ on it (plus the original two cards).

Initially you only have one card, labelled $(3, 17)$. Which of the following cards can you get?

(a) $(87, 101)$;
(b) $(20, 27)$;
(c) $(7, 13)$.

Can you describe all cards obtainable from the initial card $(3, 17)$?