The last part of our course has been a tour through Julia sets and the Mandelbrot set.

1. Julia sets

Definition 1.1. A subset $S$ of $\mathbb{C}$ is called bounded if there is a real number $R$ such that $|z| < R$ for all $z \in S$.

Definition 1.2. Let $f: \mathbb{C} \to \mathbb{C}$ be a complex function.

(a) The filled Julia set of $f$ is the set of all $z$'s whose orbit is bounded.

(b) The Julia set of $f$ is the boundary of the filled Julia set.

We never learned exactly what the ‘boundary’ of a set is, although you may remember the definition from calculus classes. For us it’s not that important, since we will always use the filled Julia set and never the Julia set itself.

We will restrict our study to the quadratic family of functions $Q_c(z) = z^2 + c$. We let $K_c$ denote the filled Julia set of $Q_c$. The following are two results we proved in class:

Lemma 1.3. If $|z| > 2$ and $|z| \geq c$, then the orbit of $z$ goes to infinity—i.e., $\lim_{n \to \infty} Q^n_c(z) = \infty$.

Corollary 1.4. Suppose $|c| < 2$. If $|Q^n_c(z)| > 2$ for some value of $n$, then the orbit of $z$ goes to infinity.

We needed these results because they lead to the following algorithm for computing the Julia set, when $|c| < 2$:

1. Pick a point $z$ in $\mathbb{C}$, and compute the first 100 iterations under the function $Q_c$.
2. If one of these iterates has magnitude larger than 2, then stop—the orbit of $z$ goes to infinity, and $z$ is not in the filled Julia set. We assign the point $z$ a color based on how many iterates it took to get a magnitude bigger than 2: red=very few, yellow=more, blue=even more, etc.
3. If none of the first 100 iterates of $z$ had magnitude larger than 2, we will make a guess that the iterates will never have magnitude larger than 2—so the orbit of $z$ is bounded, and $z$ is in the filled Julia set. We color the point $z$ black.

The above algorithm is not foolproof: if the first 100 iterates never have magnitude bigger than 2, it might still be true that the 200th iterate will have magnitude bigger than 2. So we never know with absolute certainty if a point really is in the Julia set. Said differently, the algorithm really only gives an approximation to the Julia set. By increasing the number of iterations, we get better approximations.

2. The Mandelbrot set

Theorem 2.1 (The Fundamental Dichotomy). For any fixed value of $c$, exactly one of the following two statements holds. Either

(a) The filled Julia set $K_c$ is connected, or

(b) The filled Julia set has infinitely many pieces, and does not contain any disks inside of it (in fact it is basically a Cantor set).

Definition 2.2. The Mandelbrot set $\mathcal{M}$ is defined to be $\{c \mid K_c$ is connected$\}$. In words, it is the collection of all $c$ values for which the corresponding filled Julia set is connected.
How can we use a computer to tell when a filled Julia set is connected or not? If we had to compute the whole Julia set, this would be a pain. An easier way is provided by the next result:

**Theorem 2.3.** If the orbit of 0 under $Q_c$ is bounded, then $K_c$ is connected. If the orbit of 0 is unbounded, then $Q_c$ has infinitely many pieces.

We didn’t prove Theorems 2.1 and 2.3 in this course, but you should know their statements. The latter gives us an algorithm for computing the Mandelbrot set:

1. Pick a $c$ value, and compute the first 100 iterates of 0 under $Q_c$.
2. If at some point the orbit of 0 has magnitude greater than 2, then the orbit will go to infinity (by Corollary 1.4). Theorem 2.3 then says that the filled Julia set $K_c$ is not connected, and so this value of $c$ is not in the Mandelbrot set. Assign it a color based on how many iterations it took for the orbit of 0 to have magnitude greater than 2.
3. If the first 100 iterates of 0 never have magnitude larger than 2, we will guess that this is true for all the iterates. In this case the orbit of 0 is bounded, so by Theorem 2.3 the filled Julia set is connected—that is, $c$ is in the Mandelbrot set. We color the point black.

Again, the algorithm is not foolproof—it really only gives us an approximation to the Mandelbrot set. We can improve the approximation by increasing the number of iterations we perform.

### 3. Decorations in the Mandelbrot set

In order to understand the Mandelbrot set, we break it up into different pieces. We can first look for all $c$ values such that $Q_c$ has an attractive fixed point. This turns out to be the main cardioid in the Mandelbrot set. Then we will look for $c$ values where $Q_c$ has an attractive two-cycle, or an attractive three-cycle, and so on. It is a (hard) theorem that whenever you have an attractive cycle, the orbit of 0 will necessarily converge to it—this is because 0 is the only critical point of $Q_c$. So the presence of an attractive cycle forces the orbit of 0 to be bounded, and therefore forces $c$ to be in the Mandelbrot set.

#### 3.1. The main cardioid

To have a fixed point, we need $Q_c(z) = z$. That is, $z^2 + c = z$. Re-write this as $z^2 - z + c = 0$, and the solutions are

$$p_\pm = \frac{1 \pm \sqrt{1 - 4c}}{2}.$$  

For one of these to be attractive, we need either $|Q'_c(p_-)| < 1$ or $|Q'_c(p_+)| < 1$. Since $Q'_c(z) = 2z$, this says that we want

$$|1 \pm \sqrt{1 - 4c}| < 1$$

(to be clear: we want the inequality to hold for + OR -). Let $w = 1 - 4c$, and write $w = re^{i\theta} = r \cos \theta + (r \sin \theta)i$. Then $\sqrt{w} = \sqrt{r}e^{i\theta/2}$, and our inequality becomes

$$\left(1 \pm \sqrt{\cos\left(\frac{\theta}{2}\right)}\right)^2 + \left(\sqrt{\sin\left(\frac{\theta}{2}\right)}\right)^2 < 1.$$ 

This simplifies to $1 \pm 2\sqrt{\cos\left(\frac{\theta}{2}\right)} + r < 1$, or $\sqrt{r} < \mp 2\cos\left(\frac{\theta}{2}\right)$. Finally, squaring both sides gives

$$r < 4 \cos^2\left(\frac{\theta}{2}\right).$$

This is the equation for a cardioid, which you can easily sketch. It intersects the real axis at 0 and 4, with the cusp of the cardioid at 0.

Finally, remember that this cardioid graphs the possibilities for $w$ (since $w = re^{i\theta}$). If we remember that $w = 1 - 4c$, or $c = (1 - w)/4$, then we find that the $c$ values are described by a cardioid which intersects the real axis at 1/4 and $-3/4$, with the cusp at 1/4.
3.2. The case of attractive two-cycles. We next look for all \( c \) values for which \( Q_c \) has an attractive two-cycle. To find the two-cycles, we need to solve \((z^2 + c)^2 + c = z\). This is a degree 4 equation, but we already know two of the solutions—namely, the fixed points \( p_+ \) and \( p_- \) we already found. Dividing the polynomial \( z^4 + 2cz^2 - z + (c^2 + c) \) by the polynomial \( z^2 - z + c \) (the latter of which gives the equation for the fixed points), we get \( z^2 + z + (c + 1) \). So the points on the two-cycles are the solutions to \( z^2 + z + (c + 1) = 0 \). These are

\[
q_\pm = \frac{-1 \pm \sqrt{1 - 4(c + 1)}}{2}.
\]

To ensure that this is an attractive two-cycle, we need to require that

\[
|Q'_c(q_-)| \cdot |Q'_c(q_+)| < 1.
\]

This becomes the inequality \(|(-1 + \sqrt{1 - 4(c + 1)})(-1 - \sqrt{1 + 4(c + 1)})| < 1\). Multiplying out, we find

\[
|4(c + 1)| < 1
\]

or

\[
|c + 1| < \frac{1}{4}.
\]

The \( c \)-values satisfying this inequality form a disk with radius \( \frac{1}{4} \), centered at \(-1\).

3.3. Attractive cycles of other periods. It is harder to exactly describe all \( c \) values for which \( Q_c \) has an attractive 3-cycle, but we can find the centers for these regions without working too hard. We look for the \( c \) values which give an attractive 3-cycle containing \( 0 \). The orbit of \( 0 \) looks like

\[
0 \rightarrow c \rightarrow c^2 + c \rightarrow (c^2 + c)^2 + c
\]

and for this to be a 3-cycle we need

\[
(c^2 + c)^2 + c = 0.
\]

Notice that this is a degree 4 equation, and so will have four complex solutions. Mathematica can find them numerically for us. It gives: \( c = 0 \), \( c = -1.75488 \), \( c = -0.1226 \pm 0.7449i \). The first solution, \( c = 0 \), is the center for the period 1 part of the Mandelbrot set (the main cardioid). The remaining three \( c \)-values are the centers for the three period 3 decorations in the Mandelbrot set.

Exercise 3.4. Using a computer program that allows you to zoom in on pieces of the Mandelbrot set, locate the three period 3 decorations.

To find the centers of the period 4 decorations, we make \( 0 \) lie on a cycle of period 4. For this, \( c \) needs to satisfy the equation

\[
((c^2 + c)^2 + c)^2 + c = 0.
\]

This is a degree 8 equation, and so has eight complex solutions. One of them corresponds to the period 1 decoration (the main cardioid), and one of them corresponds to the period 2 decoration. This leaves six decorations which have period 4. Mathematica gives them as: \( c = -1.3107 \), \( c = -1.9408 \), \( c = -0.1565 \pm 1.03325i \), \( c = 0.28227 \pm 0.53006i \).

Exercise 3.5. Repeat the previous exercise, but find the six decorations which have period 4.

For period 5 decorations we would solve a degree 16 equation. One solution corresponds to the period 1 decoration, leaving 15 decorations which have period 5.

For period 6 decorations we would solve a degree 32 equation. One solution corresponds to period 1, one solution corresponds to period 2, and three solutions correspond to period 3 (found above). This leaves 27 period 6 decorations.

You get the idea.
3.6. **Bulbs around the main cardioid.** Each bulb coming off the main cardioid can be labelled with a rational number \( \frac{p}{q} \). The denominator is the *period* of the bulb, which can be identified in any of the following ways:

1. The period is the number of spokes, or ‘antennas’, near the top of the bulb (including the trunk which attaches to the main part of the bulb).
2. Looking in the Julia set for any point in the bulb, the period is the number of ‘lobes’ in a cluster.
3. Given any \( c \) in the bulb, and any point in the corresponding Julia set, it is the period of the cycle that the orbit converges to.

The numerator \( p \) can be identified in either of the following ways:

1. Look at the antennas near the top of the bulb. Starting from the main trunk and moving counterclockwise, count which antenna is the smallest. The number of this antenna is \( p \) (so if the second antenna is the smallest, \( p = 2 \)).
2. Choose a \( c \) value in the bulb, compute the Julia set, and find the resulting attractive cycle. This cycle will jump from one lobe of the Julia set to another. Starting in one lobe, it will move counterclockwise—possibly skipping over other lobes in the process. The number \( p \) is the number of lobes you pass when going from one point of the cycle to the next.

We let \( B_{p/q} \) denote the bulb corresponding to \( \frac{p}{q} \) via the above rules. The bulbs on the main cardioid obey the following pattern:

**Theorem 3.7.** Pick two bulbs \( B_{p/q} \) and \( B_{p'/q'} \). The largest bulb between them is \( B_{(p+p')/(q+q')} \).

The **Farey sum** (also called the **Farey child**) of \( \frac{p}{q} \) and \( \frac{p'}{q'} \) is defined to be

\[
\frac{p}{q} \oplus \frac{p'}{q'} = \frac{p + p'}{q + q'}.
\]

We can make sequences of rational numbers by starting with \( \{ \frac{0}{1}, \frac{1}{1} \} \) and constructing all their Farey progeny. After the first generation we get

\[
\begin{array}{cccc}
0 & 1 & 1 & 1 \\
1 & 2 & 1 & 1
\end{array}
\]

then after the second generation we have

\[
\begin{array}{ccccccc}
0 & 1 & 1 & 2 & 1 & 1 & 1 \\
1 & 2 & 3 & 1 & 1 & 1 & 1
\end{array}
\]

Generation three gives

\[
\begin{array}{cccccccccccc}
0 & 1 & 1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 1 & 1 \\
1 & 2 & 3 & 2 & 3 & 1 & 4 & 3 & 4 & 1 & 1 & 1
\end{array}
\]

And so on. These sequences are called **Farey sequences**. The above theorem says that if you start at the cusp of the main cardioid of the Mandelbrot set and move around counterclockwise, the patterns of the bulbs form a Farey sequence. The size of the bulbs is controlled by which generation they correspond to in the Farey sequences.

**Exercise 3.8.**

(a) Draw a picture of the main cardioid of the Mandelbrot set, and label all the period 2, 3, and 4 bulbs attached to it. Also label the bulb \( B_{3/8} \).

(b) What equation would you need to solve in order to find all the centers of the period 8 decorations?

(c) How many period 8 decorations are there?

(d) Suppose I gave you a picture of the Mandelbrot set, together with three marked points lying in either the main cardioid or adjoining bulbs. I also give you three pictures of Julia sets. You should be able to match each Julia set to the point in the Mandelbrot set with which it corresponds.

(e) Explain the algorithm we use to compute the Mandelbrot set.
3.9. Bifurcation points. Each of the bulbs attached to the main cardioid is attached in exactly one point (if it doesn’t look like this in a certain picture of the Mandelbrot set, it’s because the picture was drawn using an algorithm with too few iterations). In the first part of the course we already studied the point where the period two bulb is attached—this is when $c = -\frac{3}{4}$, and it is a period doubling bifurcation. The other points of attachment occur where an attractive fixed point becomes neutral and then splits into an attractive cycle. The point where $B_{1/3}$ is attached is a period tripling bifurcation, the point where $B_{2/5}$ is attached is a period quintupling bifurcation, etc.

We have period scaling bifurcations of all orders.

Even without using computers, we can find out where these bifurcation points occur. We first recall that our fixed points are

$$p_{\pm} = \frac{1 \pm \sqrt{1 - 4c}}{2}.$$  

We let $w = 1 - 4c$ and write $w$ in polar form as $w = re^{i\theta}$. We then have

$$p_{\pm} = \frac{1 \pm \sqrt{r}e^{i\theta/2}}{2} = \frac{1 \pm \sqrt{r} \cos(\theta/2) \pm \sqrt{r} \sin(\theta/2)}{2} \cdot i.$$  

Only one of these fixed points will be attractive. To understand which one, we need to be careful about the signs. So let’s use the notation

$$p_u = \frac{1 \pm u\sqrt{r}e^{i\theta/2}}{2} = \frac{1 \pm u\sqrt{r} \cos(\theta/2) \pm u \sqrt{r} \sin(\theta/2)}{2} \cdot i,$$

where $u$ is either $+1$ or $-1$. We then have that

$$|Q'_u(p_u)|^2 = (1 + u\sqrt{r} \cos(\theta/2))^2 + (u\sqrt{r} \sin(\theta/2))^2 = 1 + 2u\sqrt{r} \cos(\theta/2) + r.$$

For an attractive fixed point we need $|Q'_u(p_u)| < 1$, and since $r$ is positive this will only occur when $u \cos(\theta/2)$ is negative. So we have the following conclusion:

The attractive fixed point is $p_u$ where $u$ is $+1$ or $-1$ according to the rule that $u \cos(\theta/2)$ must be negative.

Now, there are two things we know about the $c$-value where the bifurcation point for $B_{p/q}$ occurs. These are:

1. $r = 4 \cos^2(\theta/2)$
2. $Q'_u(p_u) = e^{2\pi(p/q)i}$.

The first comes from the fact that $c$ must be on the boundary (the outer rim) of the main cardioid (recall from previous discussion that the equation for the main cardioid is $r < 4 \cos^2(\theta/2)$). The second comes from discussion in class.

Plugging our equation for $p_u$ into equation (2), we get

$$\cos(2\pi(p/q)) = 1 + u\sqrt{r} \cos(\theta/2) \quad \text{and} \quad \sin(2\pi(p/q)) = u\sqrt{r} \sin(\theta/2).$$

Equation (1) says that $\sqrt{r} = \pm 2 \cos(\theta/2)$, but we can be more precise about the sign. Since $\sqrt{r}$ is necessarily positive, the sign must be $-u$ (since $u$ is the sign which makes $u \cos(\theta/2)$ negative). So $\sqrt{r} = -2u \cos(\theta/2)$. Substituting this into the two equations above, we get

- $\cos(2\pi(p/q)) = 1 - 2 \cos^2(\theta/2) = -\cos(\theta)$
- $\sin(2\pi(p/q)) = -2 \cos(\theta/2) \sin(\theta/2) = -\sin(\theta)$.

We have used two trig identities: $\sin(2x) = 2 \sin(x) \cos(x)$ and $\cos(2x) = 1 - 2 \cos^2(x)$. These two equations completely determine $\theta$.

The solutions to $\sin(\theta) = -\sin(2\pi(p/q))$ are $\pi(1 + 2\frac{p}{q})$ and $\pi(2 - 2\frac{p}{q})$. The only one of these which satisfies the equation $\cos(2\pi(p/q)) = \cos(\theta)$ is $\pi(1 + 2\frac{p}{q})$. So $\theta = \pi(1 + 2\frac{p}{q})$.

What does this mean? It means that $w$ lies on the ray of complex numbers whose argument is $\pi + 2\pi(p/q)$. These are rays emanating from the cusp of the main cardioid. Translating to $c$, we
find that \( c \) lies on the ray of complex numbers emanating from the cusp of the main cardioid which is an angle of \( \pi - 2\pi(p/q) \) radians measured clockwise from the negative real axis.

Summary: When \( \frac{p}{q} \leq \frac{1}{2} \), the bulb \( B_{p/q} \) is attached to the main cardioid at a bifurcation point characterized as follows. If you draw the segment from the cusp of the main cardioid to this bifurcation point, the angle from the segment to the negative real axis measures \( \pi - 2\pi(p/q) \) radians.

4. The anatomy of the filled Julia set

4.1. Understanding the Julia sets through codes. Fix a \( c \in \mathbb{C} \). If \( z \) lies on a cycle for \( Q_c \), then the orbit of \( z \) is certainly bounded, and therefore \( z \) is in \( K_c \). So all periodic points of \( Q_c \), and all eventually periodic points for \( Q_c \), lie in the filled Julia set \( K_c \). Our goal is to find them in the computer-generated picture of \( K_c \).

In class we invented a code that allows us to partially understand the filled Julia set when \( c \) belongs to \( B_{1/3} \). Around one of the fixed points of \( Q_c \) are three “lobes” which we labelled \( B, M, \) and \( L \) (for Big, Medium, and Little). The \( B \)-lobe is the one that contains the point 0. Recall that the Julia set has a symmetry: if \( z \) is in \( K_c \) then \( -z \) is also in \( K_c \). This is a 180-degree rotation symmetry about the point 0.

We noted that applying \( Q_c \) maps the \( L \)-lobe entirely into the \( M \)-lobe, the \( M \)-lobe entirely into the \( B \)-lobe, and it does something complicated on the \( B \)-lobe. To explain what this is, we also need names for the biggest lobes emanating from the sides of the \( B \)-lobe: we called these \( S_{MM}, S_{ML}, S_{LM}, \) and \( S_{LL} \).

A sequence like \( BMS_{ML}LM \ldots \) can be thought of as describing areas (or points) of \( K_c \). In this case we start at the central fixed point, move through the \( B \)-lobe to the opposite vertex, then through the attached \( M \)-lobe into its attached \( S_{ML} \)-lobe; from there we move across to the attached vertex and then into the \( L \)-lobe, then across into the \( M \)-lobe, and so forth.

We know that
\[
Q_c(La_1a_2a_3\cdots) = Ma_1a_2a_3\ldots \quad \text{and} \quad Q_c(Ma_1a_2a_3\cdots) = Ba_1a_2a_3\ldots.
\]

Exercise 4.2. Determine the corresponding rules for
(i) \( Q_c(BMa_1a_2\ldots) \),
(ii) \( Q_c(BLa_1a_2\ldots) \),
(iii) \( Q_c(BS_{MM}a_1a_2\ldots) \),
(iv) \( Q_c(BS_{ML}a_1a_2\ldots) \),
(v) \( Q_c(BS_{LM}a_1a_2\ldots) \),
(vi) \( Q_c(BS_{LL}a_1a_2\ldots) \).

[Hint: Recall that \( z \) and \( -z \) have the same image under \( Q_c \).]

Exercise 4.3. Using the rules from the previous exercise, determine the code for the second fixed point of \( Q_c \). Also determine the codes for the points on the 2-cycle of \( Q_c \), and for the points on the non-attractive 3-cycle of \( Q_c \). Find these in the computer-generated picture of the Julia set.

Exercise 4.4. Let \( c \) belong to the bulb \( B_{1/3} \). Let \( p \) denote the fixed point of \( Q_c \) that lies at the center of the three main lobes. In the picture of the Julia set, find all the points
(i) which go to \( p \) after one iteration (there is only 1, besides \( p \) itself);
(ii) that go to \( p \) after two iterations (there are 2 of them in addition to what you have already found);
(iii) that go to \( p \) after three iterations (there are four of these).

4.5. External rays for the filled Julia set. For each \( k \in [0, 1] \) there is a curve which starts on the boundary of the filled Julia set and goes out towards infinity, and as you go out to infinity the points on this curve have the form \( re^{i\theta} \) where \( \theta = 2\pi \cdot k \). These curves are called “external rays”
(even though they are not straight like a ray usually is), and they will be denoted $R_k$. When you apply $Q_c$, the ray $R_k$ becomes $R_{D(k)}$, where $D(k)$ is the doubling function.

Let $c$ belong to the bulb $B_{1/3}$, and let $p$ denote the fixed point of $Q_c$ that lies at the center of the three main lobes. The function $Q_c$ will map each ray to one of the others, and so these rays have the form $R_k$, $R_l$, $R_m$, where \( \{k, l, m\} \) is a 3-cycle for the doubling function. There are only two such 3-cycles: they are \( \{1/7, 2/7, 4/7\} \) and \( \{3/7, 6/7, 5/7\} \). By looking at the geometry of the picture, we find that \( \{k, l, m\} \) must be the former (the other values correspond to the external rays converging to the non-attractive 3-cycle of $Q_c$).

**Exercise 4.6.**

(i) Find the three external rays that converge to the point $-p$.

(ii) Let $q_1$ and $q_2$ be the two points in the Julia set which hit $p$ after exactly two iterations. Find the three external rays that converge to each of these points.

For each “pinch point” in the Julia set, one can find the three external rays that converge there. This gives us a kind of map of the Julia set.

Let \( \{a, b\} \) be the 2-cycle for $Q_c$. Because $Q_c(Q_c(a)) = a$, the ray $R_k$ that converges to $a$ must have $Q_c(Q_c(R_k)) = R_k$. Therefore $D(D(k)) = k$; that is, $k$ is a period 2 point for the doubling function. We can find all such period 2 points: they are $1/3$ and $2/3$. Therefore the rays converging to $a$ and $b$ are $R_{1/3}$ and $R_{2/3}$. By looking at our “map” of the external rays of the Julia set, we can find where $R_{1/3}$ and $R_{2/3}$ are and therefore find $a$ and $b$. 