SPECIAL RELATIVITY

1. Minkowski space

1.1. **Spacetime.** Suppose we are studying a particle moving through space. A standard way to model this situation mathematically is via a function $\gamma\colon \mathbb{R} \to \mathbb{R}^3$, where $\gamma(t)$ is the location of the particle at time t. In this set-up, time and space are noticeably separated: the domain of γ is the set of all possible times, and the target of γ is the set of all possible spatial coordinates.

In special relativity it will be important to have time and space appear as part of the same mathematical object. We will use the vector space $\mathbb{R}^4$, consisting of all vectors (x, y, z, t). Physically, a vector in $\mathbb{R}^4$ corresponds to an 'event' in the universe—something which occurs at a particular point in both space and time. The vector space $\mathbb{R}^4$ will be called *spacetime* (or the 'spacetime continuum' if you want to sound like you're in a science fiction movie).

Let's be more precise. In physics we are concerned with performing experiments and predicting their outcomes. We use 'experiment' in the broadest possible sense, here— meaning anything where we want to predict what happens. So our experiment might be a person getting inside a spaceship and travelling to the moon, for instance. In our laboratory (again intended in a rather broad sense, so that for example it might be the whole state of Oregon) we fix once and for all an origin where we set $x = y = z = 0$, three perpendicular axes in space, and a certain time which will be called $t = 0$. We decide which units we'll use to measure distance and time, for instance meters and seconds. Every vector in $\mathbb{R}^4$ can now be identified with a precise location in time and space.

As an example, imagine the room you're in as your laboratory, and pick one corner to be the origin; decide which axes correspond to x, y, and z; and say that '$t = 0$' corresponds to midnight GMT on January 1, 2004. Every event that ever occurs or has occurred in the room can be specified by giving (x, y, z) and t. If I tell you that 30 seconds after the new year a firework exploded at coordinates $(10, 15, 31)$, you know what I mean. The firework exploding corresponds to the vector $(10, 15, 31, 30) \in \mathbb{R}^4$. If another firework exploded in the same location but 30 seconds *before* the new year, it corresponds to $(10, 15, 31, -30) \in \mathbb{R}^4$.

1.2. **The Minkowski inner product.** When doing Euclidean geometry, we use the dot product on $\mathbb{R}^4$ given by

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \cdot \begin{bmatrix} x' \\ y' \\ z' \\ t' \end{bmatrix} = xx' + yy' + zz' + tt'.$$

Recall that this definition doesn't just come out of the sky: it comes from the fact that the two vectors span a plane, give two sides of a triangle lying in that plane, and from applying the ordinary law of cosines to this triangle. We have to ask ourselves, however, whether this has any meaning when dealing with the physics of spacetime—there's certainly no reason to believe that it does. In fact the dot product feels fishy here: note that the terms xx', yy', and zz' will all be measured in meters^2, whereas tt' is in sec^2. Adding these together seems like adding apples and oranges.

In special relativity it turns out that we *don't* use the dot product on $\mathbb{R}^4$; we use something else called the **Minkowski inner product**. We'll denote this by $\langle \mathbf{v}, \mathbf{w} \rangle$, so that we never confuse it with the dot product. It is defined as

$$\left\langle \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}, \begin{bmatrix} x' \\ y' \\ z' \\ t' \end{bmatrix} \right\rangle = -xx' - yy' - zz' + c^2 tt'.$$

Here c is a physical constant, which will turn out to be the speed of light in a vacuum—roughly $3 \cdot 10^8 m/s$. At this point I really *have* just pulled this definition out of the sky; I've given you no reason to believe that it is a good thing to look at, or that it has any physical significance. It will take time to understand the answers to those questions, but that is what we will work towards over the next couple of lectures.

For now, let us note that the Minkowski inner product has many of the same properties as the dot product:

(1) $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$;
(2) $\langle s\mathbf{v}, \mathbf{w} \rangle = s\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, s\mathbf{w} \rangle$;
(3) $\langle \mathbf{v_1} + \mathbf{v_2}, \mathbf{w} \rangle = \langle \mathbf{v_1}, \mathbf{w} \rangle + \langle \mathbf{v_2}, \mathbf{w} \rangle$.

However it does *not* have the property that $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ for all vectors $\mathbf{v}$. For instance $\langle \mathbf{e}_i, \mathbf{e}_i \rangle = -1 < 0$ for $i = 1, 2, 3$, whereas $\langle \mathbf{e}_4, \mathbf{e}_4 \rangle = c^2 > 0$. This turns out to be a good thing, for reasons we'll now try to explain.

One fundamental fact in special relativity is that information cannot be relayed at faster than the speed of light. Consider a light pulse sent at $(0, 0, 0, 0)$, directed along the positive x-axis. So in 1 second the light pulse arrives at $(c, 0, 0, 1)$, in 2 seconds it is at $(2c, 0, 0, 2)$, etc. Let's say that a person is standing at spatial coordinates $(c, 0, 0)$, and that when he sees the light pulse he trips and falls—his fall is an event at $(c, 0, 0, 1)$. The emission of the light pulse at $(0, 0, 0, 0)$ influenced the event of the person falling at $(c, 0, 0, 1)$.

Note, however, that if the person had fallen at $(c, 0, 0, 0.5)$, this would have happened half a second *before* the light pulse reached him. In fact, no information regarding the event at $(0, 0, 0, 0)$ could possibly have reached coordinates $(c, 0, 0)$ by time $t = 0.5$—since no information can be translated faster than the speed of light—and so in this case the emission of the light pulse and the person falling cannot possibly be related.

As a final case, consider if the person had fallen at $(c, 0, 0, 1.2)$. This happened after the light pulse arrived at $(c, 0, 0)$, and so could possibly have been influenced by it.

Note that $\left\langle \begin{bmatrix} c \\ 0 \\ 0 \\ 1.2 \end{bmatrix}, \begin{bmatrix} c \\ 0 \\ 0 \\ 1.2 \end{bmatrix} \right\rangle = -c^2 + (1.2)^2 c^2 > 0,$

whereas

$$\left\langle \begin{bmatrix} c \\ 0 \\ 0 \\ 0.5 \end{bmatrix}, \begin{bmatrix} c \\ 0 \\ 0 \\ 0.5 \end{bmatrix} \right\rangle = -c^2 + (0.5)^2 c^2 < 0 \quad \text{and} \quad \left\langle \begin{bmatrix} c \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} c \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle = -c^2 + c^2 = 0.$$

Briefly, if $\langle \mathbf{v}, \mathbf{v} \rangle < 0$ it means that an event at the origin couldn't possibly influence an event at $\mathbf{v}$, or the other way around. If $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ it means that one of the events could possibly influence the other, and if $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ it means that one of the events can influence the other only if information is relayed at exactly the speed of light.

Definition 1.3. *Let* $\mathbf{v}$ *be a vector in* $\mathbb{R}^4$.

(a) $\mathbf{v}$ *is called* ***timelike*** *if* $\langle \mathbf{v}, \mathbf{v} \rangle > 0$*, in which case the* ***relativistic length*** *of* $\mathbf{v}$ *is defined to be* $\sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.

(b) $\mathbf{v}$ *is called* ***spacelike*** *if* $\langle \mathbf{v}, \mathbf{v} \rangle < 0$*, in which case the* ***relativistic length*** *of* $\mathbf{v}$ *is defined to be* $\sqrt{-\langle \mathbf{v}, \mathbf{v} \rangle}$.

(c) $\mathbf{v}$ *is called* ***lightlike*** *(or* ***null****) if* $\langle \mathbf{v}, \mathbf{v} \rangle = 0$*, in which case the* ***relativistic length*** *of* $\mathbf{v}$ *is defined to be* $\sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = 0$.

If we have two events, corresponding to vectors $\mathbf{v}$ and $\mathbf{w}$, we can compare them by shifting one of them to the origin: so we look at $\mathbf{0}$ and $\mathbf{v} - \mathbf{w}$ instead, or $\mathbf{0}$ and $\mathbf{w} - \mathbf{v}$. So the **relativistic distance** between two vectors $\mathbf{v}$ and $\mathbf{w}$ is defined to be the relativistic length of $\mathbf{v} - \mathbf{w}$. If $\langle \mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{w} \rangle < 0$, then an event at $\mathbf{v}$ cannot possibly influence an event at $\mathbf{w}$, or vice versa. If $\langle \mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{w} \rangle > 0$ then an event at $\mathbf{v}$ *could* influence an event at $\mathbf{w}$, or the other way around. Finally if $\langle \mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{w} \rangle = 0$ then an event at $\mathbf{v}$ can influence an event at $\mathbf{w}$ only if information is relayed at exactly the speed of light—nothing slower would suffice.

2. Relativity and coordinate changes

Let us imagine a person travelling on a high-speed train, at constant velocity with respect to the earth. He chooses a coordinate system, allowing him to model the universe by $\mathbb{R}^4$ as we discussed in the last section. But suppose there is another person, travelling on another high-speed train, also at constant velocity with respect to the earth. He chooses his own coordinate system, and has his *own* way of modelling the universe by $\mathbb{R}^4$. These two experimenters each perform their own measurements and develop their own theories. If we are to believe that the universe is a reasonable place, we should expect that one could convert all of the first experimenter's observations/theories into the second one's coordinate system, and that everything would still make sense.

Let's cut the problem down to size a little. Imagine the two trains are moving on parallel tracks very close together, and call the two people on the trains 'Bart' and 'Homer'. Imagine Bart is moving from west to east, Homer is moving from east to west, and they are moving with constant velocity v with respect to each other. From Bart's perspective, he is standing still and Homer is moving. From Homer's perspective, *he* is the one standing still and Bart is moving. We will ignore the y and z spatial coordinates for the moment, and just use the x-coordinate measuring distance along the railway. Bart and Homer agree (in advance) to call the place where they cross $x = 0$, and at the moment they cross they will both reset their clocks to $t = 0$.

Bart's model for the universe is $\mathbb{R}^2$ (with vectors (x,t)), but we'll call it $\mathbb{R}^2_B$ to avoid confusion. We'll call Homer's model for the universe $\mathbb{R}^2_H$. We are looking for a function $L\colon \mathbb{R}^2_B \to \mathbb{R}^2_H$ which transforms from Bart's model to Homer's model. The *basic principle of relativity* says: L is a linear transformation with the property that

$$\langle L(\mathbf{v}), L(\mathbf{w})\rangle = \langle \mathbf{v}, \mathbf{w}\rangle$$

for all vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2_B$. Another way of saying this is that L must preserve the relativistic distance of every vector, together with the properties of being timelike, spacelike, or lightlike.

Of course we know L is represented by a matrix $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$. Some physical intuition shows that $p > 0$, $r > 0$, $s > 0$, and $-pv + q = 0$. We also need to choose the entries so that the following equations hold:

(1) $\langle L(e_1), L(e_1)\rangle = \langle e_1, e_1\rangle$
(2) $\langle L(e_2), L(e_2)\rangle = \langle e_2, e_2\rangle$
(3) $\langle L(e_1), L(e_2)\rangle = \langle e_1, e_2\rangle$.

Using the definition of the Minkowski inner product, these become:

(1') $-p^2 + c^2r^2 = -1$
(2') $-q^2 + c^2s^2 = c^2$
(3') $-pq + c^2rs = 0$.

The first two equations show that

$$r = \frac{\sqrt{p^2-1}}{c}, \qquad s = \frac{\sqrt{q^2+c^2}}{c}.$$

Plugging into the third equation gives $pq = \sqrt{p^2-1}\sqrt{q^2+c^2}$. Squaring gives $p^2q^2 = p^2q^2 - q^2 + c^2p^2 - c^2$, or $q^2 = c^2(p^2-1)$. Substituting $q = pv$ and rearranging, we find that

$$p^2 = \frac{c^2}{c^2-v^2} = \frac{1}{1-(\frac{v}{c})^2}$$

or that

$$p = \sqrt{\frac{1}{1-(v/c)^2}}.$$

Substituting into the formulas for r and s, we find $r = pv/c^2$ and $s = p$. Finally, putting everything together, we've found that the matrix for L has the form

$$\begin{bmatrix} p & pv \\ \frac{pv}{c^2} & p \end{bmatrix}$$

where $p = \sqrt{\frac{1}{1-(v/c)^2}}$.

Example 2.1. Suppose the relative speed of Bart and Homer is half the speed of light—i.e., $v = \frac{c}{2}$. Then the transformation L has matrix

$$\begin{bmatrix} 2/\sqrt{3} & c/\sqrt{3} \\ 1/c\sqrt{3} & 2/\sqrt{3} \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 2 & c \\ 1/c & 2 \end{bmatrix}.$$