Analysis of a water-propelled rocket: A problem in honors physics

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The air-pumped, water-propelled rocket is a common child's toy, yet forms a reasonably complicated system when carefully analyzed. A lab based on this system was included as the final laboratory project in the honors version of General Physics I at the USAF Academy. The numerical solution for the height of the rocket is presented, as well as several analytic approximations. Five out of six lab groups predicted the maximum height of the rocket within experimental error.

I. INTRODUCTION

The study of rocket motion has been used for decades to excite students with the study of physics. (See Refs. 1–4, for example.) Combined with the use of electronic computers, students can begin to solve many interesting, ''real world'' problems. In the honors versions of a calculus-based introductory mechanics course, I assigned my students the problem of analyzing the motion of an air-pumped, water-propelled rocket. The final goal was to determine the optimum amount of water to put into the rocket in order to achieve the maximum possible height. While we used small toy rockets, most of this analysis would also apply to the popular demonstration using 2-ℓ soda bottles pressurized by a bicycle pump.

II. MECHANICS OF ROCKET MOTION

There are numerous references to the basic physics of rockets. In addition to those listed above, the reader may consult almost any university physics text. The basic problem is to find the thrust, drag, and mass of the rocket as a function of time in order to find the acceleration, velocity, and position. The following sections develop the differential equations to be solved numerically, as well as some useful analytic approximations.

A. Thrust

The thrust, T, of a rocket due to the ejection of mass from the nozzle is

$$T=\left|v_e\frac{dM}{dt}\right|, \tag{1}$$

where v_e is the exhaust velocity of the ejected mass in the rocket's frame of reference and dM/dt is the rate at which mass is ejected from the rocket. In our case, the mass is the water that is pushed out as a result of the elevated air pressure inside the rocket. Because v_e and dM/dt both depend on the pressure inside the rocket, finding the time profile of the thrust is nontrivial. However, it is within the capability of better introductory physics students.

Bernoulli's equation (conservation of energy) is applied at two points along a streamline. This can be written generally as

$$P_1+\tfrac{1}{2}\rho v_1^2+\rho g y_1=P_2+\tfrac{1}{2}\rho v_2^2+\rho g y_2\,.$$

Figure 1 shows a schematic of the rocket. Take point 1 as the surface of the water inside the rocket and point 2 just outside the nozzle. Neglecting the pressure difference due to the height of the water and neglecting the velocity of the water at the surface compared to the velocity at the nozzle, we obtain

$$P=P_a+\tfrac{1}{2}\rho_w v_e^2, \tag{2}$$

where P is the pressure inside the rocket, P_a is atmospheric pressure, and ρ_w is the density of water. In addition to the assumptions listed above, we must also take as valid all the assumptions which apply to Bernoulli's equation (principally incompressible, nonviscous, irrotational flow).

Equation (2) can be solved for v_e and determines the exhaust velocity as a function of internal pressure, P. The other term needed to find the thrust from Eq. (1) is the mass flow rate. Since the mass flow rate is just the volume flow rate times the density of the water,

$$\frac{dM}{dt}=\rho_w\frac{dV}{dt}=\rho_w A_e v_e\,, \tag{3}$$

where A_e is the cross-sectional area of the exhaust nozzle. Combining Eqs. (1)–(3) gives

$$T=2(P-P_a)A_e\,. \tag{4}$$

Finding the thrust therefore depends on finding the pressure within the rocket as a function of time. As the rocket expels the water, the pressure and exhaust velocity drop, and thus the rate of pressure decrease drops. The solution begins with two assumptions: (1) the air in the rocket behaves as an ideal gas and (2) the air expands isothermally. (Justification for the isothermal assumption is given in Appendix A.) These assumptions allow us to write

$$PV=P_0V_0\,, \tag{5}$$

where P and V are the pressure and volume of air inside the rocket at any time before all the water is ejected and P_0 and V_0 are the initial pressure and volume of air. Solving for P and taking the derivative with respect to time

$$\frac{dP}{dt}=-\frac{P_0V_0}{V^2}\frac{dV}{dt}. \tag{6}$$

Now substituting from Eqs. (2), (3), and (5) to eliminate V, we get

$$\frac{dP}{dt}=-\frac{P^2}{P_0V_0}A_e\sqrt{\frac{2(P-P_a)}{\rho_w}}. \tag{7}$$

Equation (7) can be solved to obtain $P(t)$. The analytic solution is presented below for comparison but, because of the complexity of the result, I had the students utilize a numerical solution for $P(t)$. To solve (7) analytically, separate vari-

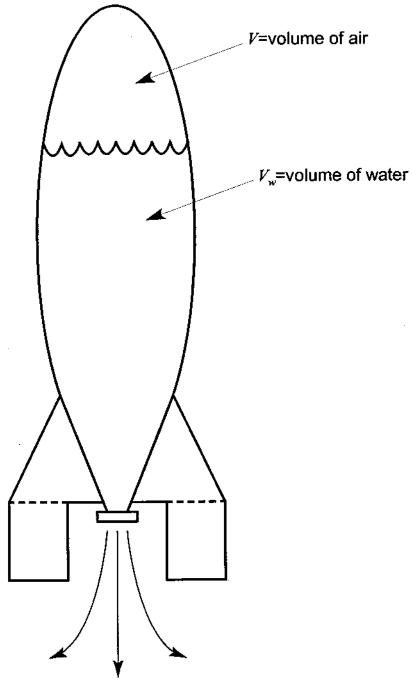

Fig. 1. Sketch of the water rocket under consideration.

ables and integrate from $t=0$ to t and from $P=P_0$ to P_f, which yields

$$t=\sqrt{\frac{\rho_w}{2}}\frac{P_0V_0}{P_aA_e}\left(\frac{\sqrt{P_f-P_a}}{P_f}-\frac{\sqrt{P_0-P_a}}{P_0}\right.$$
$$\left.+\frac{1}{\sqrt{P_a}}\left[\arctan\left(\frac{\sqrt{P_f-P_a}}{P_a}\right)-\arctan\left(\frac{\sqrt{P_0-P_a}}{P_a}\right)\right]\right). \tag{8}$$

Figure 2 shows a comparison between the numerical solution of (7) and Eq. (8). Aside from the simplicity of numerically integrating (7) compared to inverting (8) for $P(t)$, solving it numerically also allows straightforward incorporation of the next force, aerodynamic drag.

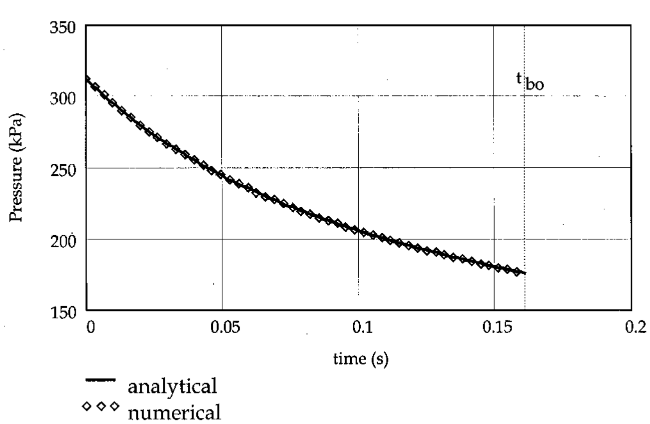

Fig. 2. Comparison of analytical result of Eq. (8) with the numerical solution from Eq. (16). Numerical solution shows only every tenth point from the numerical integration. Time of burnout is indicated by t_{bo}.

B. Drag

Aerodynamic drag is an important velocity-dependent force, but not always discussed in introductory physics. Traditionally the drag force, F_d, is expressed as

$$F_d=\tfrac{1}{2}C_dA\rho v^2, \tag{9}$$

where C_d is the drag coefficient, A is an area corresponding to the geometry of interest, and ρ is the density of air. At moderate speeds (see Appendix B), the drag coefficient is independent of the size of the object and speed of the airflow.[5] In the case of these rockets, there were two components contributing to the drag: the rocket body and the fins. Figure 1 shows that the body is roughly ellipsoidal in shape. The dashed line in Fig. 1 indicates the location of a bend in the fins to help stabilize the rocket by inducing rotation. For these rockets, drag is a fairly small effect, so the precise value of C_d is not critical. Students could go to a number of sources to obtain the necessary data to estimate C_d.[6–8] My estimates are $C_{d,\text{body}}=0.05$ and $C_{d,\text{fins}}=0.1$. The appropriate area for the body is the circular cross-section normal to the airflow. The area for the fins is the lateral area shown in Fig. 1, the surface area (of one side of each fin) which is roughly parallel to the airflow. Combining these into a single value, and using the local average air density of $\rho=1.05\,\text{kg/m}^3$ (for an elevation of 7000 ft), yields a total drag force

$$F_d=Dv^2, \tag{10}$$

where $D=2\times10^{-4}\,\text{N/(m/s)}^2$. This equation is incorporated into the numerical solution for the motion of the rocket.

Now consider the following analytic approximation for the reduction in the maximum possible height due to drag. First, drag can be neglected during the thrust phase for the following reasons: The thrust phase lasts only about 0.1–0.2 s, or about 1.5 m out of a total altitude gain of 20 m. Furthermore, drag is not the dominant force during the thrust phase (or during the coast phase, for that matter). For a speed of 20 m/s, the drag force is only about 0.08 N, compared to the thrust of 10–20 N. However, the force of gravity on the empty rocket is about 0.4 N, so drag is a minor (but significant) effect during the coast phase.

Therefore, it is possible to treat the drag force as a perturbation on the kinematic solution.[9] From kinematics (i.e., ignoring drag), the velocity profile for an initial speed v_0 is

$$v(y)=\sqrt{v_0^2-2gy}. \tag{11}$$

Now calculate the work done by the force of drag using this profile,

$$W_{\text{nc}}=\int \mathbf{F}\cdot d\mathbf{s}=-D\int_0^y v(y)^2dy=-D(v_0^2y-gy^2). \tag{12}$$

Applying this value of work in conservation of energy ($\Delta K+\Delta U=W_{\text{nc}}$) and solving for the maximum height of the rocket (m_r is the mass of the empty rocket),

$$y_{\max}=\frac{v_0^2}{2g}+\frac{m_r}{2D}\left(1-\sqrt{1+D^2v_0^4/m_r^2g^2}\right). \tag{13}$$

The fraction under the radical is just the square of the ratio of the maximum drag force to the force of gravity. For our rockets, this value was about 0.05. Therefore it can be expanded to first order. Finally, again use kinematics to replace v_0 with t, the total time of flight of the rocket from launch to impact, we obtain for the height of the rocket (to first order in the drag coefficient)

$$h=\frac{1}{8}gt^2-\frac{D}{64m_r}g^2t^4. \tag{14}$$

This expression can be used to estimate the height of the rocket given the time of flight—a much simpler measurement to make than using a sextant and trigonometry. This result will be compared to the result of numerical integration in a later section.

C. Rocket mass

Rocket ''burnout'' will be determined by one of two possible conditions: either the air expands until it forces all of the water out of the rocket or it expands until it reaches atmospheric pressure. While the latter could conceivably occur if the initial volume of the air was much smaller than the total volume inside the rocket, it is of little practical interest and is not considered further. In the former case, once all of the water is exhausted, the remainder of the air will rush out, but the air will contribute little to the thrust and is neglected.

The mass of the rocket constantly decreases until all of the water is ejected from the rocket. For a given pressure inside the rocket, the volume of the air inside the rocket was found using Eq. (5). The volume of the water, V_w, is thus the difference between the volume of the air and the total volume of the rocket, V_T. Multiplying by the density of water ($\rho_w=10^3\ \mathrm{kg/m^3}$) and adding the mass of the empty rocket ($m_r=39\ \mathrm{g}$) yields the final result for the mass of the rocket as a function of the internal pressure (before ''burnout''):

$$M(P)=\begin{cases}\rho_w(V_T-P_0V_0/P)+m_r & \text{before ``burnout''}\\ m_r & \text{after ``burnout''}.\end{cases} \tag{15}$$

This expression is used in the numerical solution.

III. NUMERICAL SOLUTION

Since I was working with second semester freshmen, they had very little experience with numerical methods. Therefore, the students used the simple Euler method (first order, forward time difference) to implement the numerical solution. The students used MATHCAD® to perform the computations. My solution was implemented using the following set of equations, combining the results of Eqs. (4), (7), (10), and (15) (the students used similar sets of equations):

$$\begin{aligned}P_{n+1}&=P_n-\frac{P_n^2}{P_0V_0}A_e\sqrt{\frac{2(P_n-P_a)}{\rho_w}}\Delta t,\\ v_{n+1}&=v_n+a(P_n,v_n)\Delta t,\\ y_{n+1}&=y_n+v_n\Delta t,\end{aligned} \tag{16}$$

where

$$a(P,v)=\frac{2A_e(P-P_a)-Dv|v|}{M(P)}-g. \tag{17}$$

Once the students had made various measurements of the rockets to determine the necessary physical parameters, only two initial conditions were left to be determined: pressure and water volume. Clearly, the higher the initial pressure, the greater the velocity of the ejected water, and hence the greater the velocity of the rocket. Therefore, I set an upper limit of 4 atm (P_a) for the pressure inside the rocket (based on destructive testing of one sample that cracked at slightly less than 5 atm). While it may have been possible to modify the pump in order to directly measure the pressure, I wanted to keep the project as simple as possible. Therefore, the pressure was calculated based on the volume of the rocket, the volume of water in the rocket, and the number of ''pumps.'' Figure 3 shows a schematic drawing of the pump used. Examining the pump, it consists of a piston moving within a cylinder. However, the pump is constructed so that there is a small amount of empty space at the end of the cylinder. As the piston is compressed, the air in the main cylinder with volume V_c (31 mL) is forced into the small space at the end of the cylinder with volume V_e (8 mL). Thus, as the rocket is pressurized, the air from the main cylinder and small space at the end (with volume V_c+V_e) is compressed into the small space and the empty space in the rocket (with volume V_e+V_0). The pressure after $j+1$ pumps, P_{j+1}, can be written:

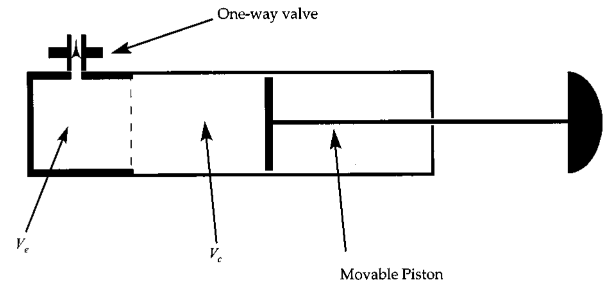

Fig. 3. Schematic diagram of the rocket pump, showing the main cylinder with volume V_c and empty space at the end of the pump with volume V_e.

$$P_{j+1}=\frac{P_jV_0+P_a(V_c+V_e)}{V_0+V_e}. \tag{18}$$

[Note that as $n\rightarrow\infty$, $P_{n+1}=P_n=P_a(V_c+V_e)/V_e\cong 5P_a$. Therefore it would appear that the rockets may have been originally designed to withstand the maximum possible pressure the pump could generate, but the rockets had degraded with age.] Thus the students could determine the number of pumps needed to achieve a pressure of $4P_a$.

Finally, the students needed only to determine the optimum water volume. They accomplished this task by repeating the calculations for a range of V_0's with the MATHCAD worksheet that they had developed. Figure 4 shows a plot of maximum height versus volume of water for $P_0=4P_a$, both including and excluding the effects of drag. Interestingly, the height is not especially sensitive to the volume of water near the maximum height (even considering that the derivative is zero at a maximum). With the modeling complete, the students were ready to launch their rockets.

IV. LAUNCHING

The students launched their rockets several times during a single class period to compare the results of their model to the actual performance of their rockets. The height was estimated in two ways. First, the students attempted to use triangulation, but the available equipment (protractors and plumb bobs) yielded less than satisfactory results because the uncertainty in the height was too large. The second technique was to measure the total time of flight for the rockets and use Eq. (14) to estimate the maximum height. Figure 5 shows

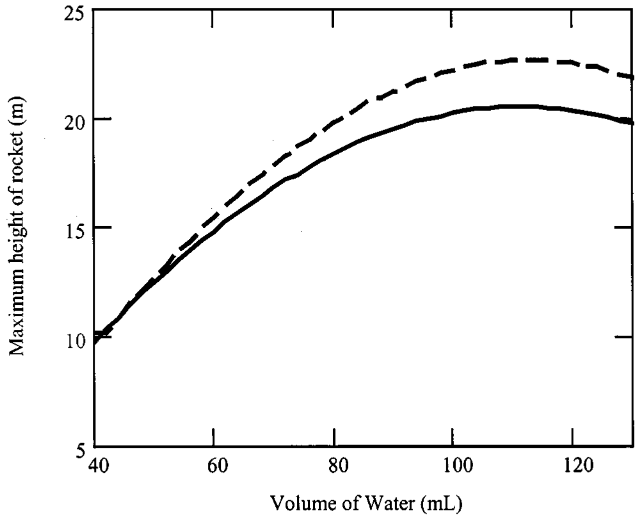

Fig. 4. Predicted height as a function of initial water volume in the rocket. The solid line shows the result of numerical integration of Eq. (16) while the dashed line shows the result neglecting drag.

maximum height versus time of flight for the analytic approximation of Eq. (14) and numerical integration of Eq. (16). The time of flight from numerical integration of (16) depended upon the initial volume of water—the point the curve doubles back indicates the time of flight corresponding to the optimum water volume. The error bars are based only on the uncertainty in measuring the time of flight (estimated to be 0.2 s), propagated through Eq. (14) using standard techniques.[10] Of the six groups, five predicted the height (as determined by time of flight) within experimental uncertainty. The only group which was not within uncertainty elected to neglect drag in their model. The success of the students as a whole is particularly significant considering the model had no free parameters!

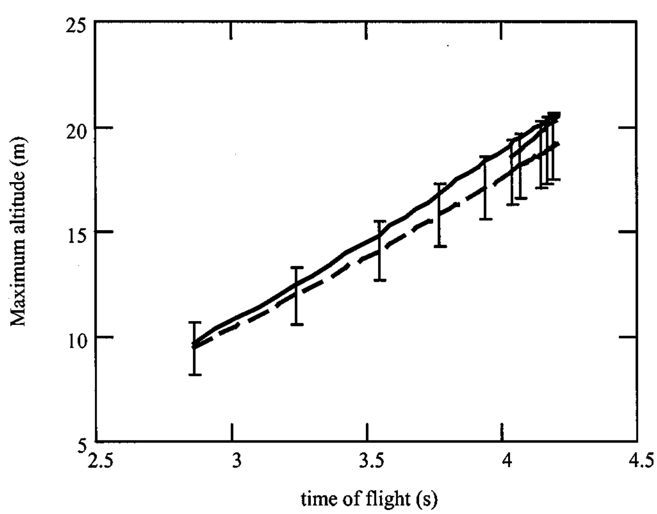

Fig. 5. Maximum altitude vs time of flight. The solid line indicates the result of numerical integration of (16) and the dashed line indicates the analytical approximation given by Eq. (14). Error bars indicate uncertainty in estimated height based on uncertainty in measuring time of flight.

V. CONCLUSIONS

Most students reacted favorably to the project. The students worked in groups of 3–4 to outline their procedures, develop the model, and predict the maximum height. Students did comment on the amount of time required, 10–20 h per group, stretched over about half the semester. However, they also commented that the ability to analyze and predict the rocket's motion was exciting and motivational. Although probably beyond the ability of many introductory students, for those willing to tackle it, they should find this a rewarding project.

APPENDIX A: JUSTIFICATION FOR ISOTHERMAL EXPANSION

In deriving Eq. (7) for the rate of change of pressure, an approximation of isothermal expansion was used. Because of the brief time involved, an adiabatic approximation would seem more natural. An adiabatic expansion will result in the air cooling as it expands, reducing the pressure and the thrust faster than an isothermal expansion. To test the sensitivity to the heat gain through contact with the walls of the rocket, one can solve the heat flow equation,

$$\nabla^2 T = -\frac{1}{\alpha}\frac{\partial T}{\partial t}. \tag{A1}$$

Here α is the ratio of the thermal conductivity to the specific heat and is taken to be approximately $2\times 10^{-4}\,\mathrm{m^2/s}$. For simplicity, treat the rocket as an infinite cylinder. Then it is straightforward to solve Eq. (A1) in one dimension (radial) with a fixed temperature at the walls of the rocket by expansion in Bessel functions.[11] For the initial condition, use the temperature change given by an adiabatic expansion, typically about 40 °C, taken uniformly across the cylinder. In 0.2 s, the temperature of the expanding air returns at least halfway to the ambient temperature 0.8 cm from the wall of the rocket. Given that (1) the rocket is not really a cylinder and (2) the air has considerable volume near the surface of the water and the front end of the rocket, then significantly less than half the volume of the rocket deviates more than 20 °C from the initial temperature. Since neither the isothermal nor the adiabatic approximations rigorously hold, I used the simpler isothermal approximation. This had an additional advantage because my course did not include a block on thermodynamics: my students had already learned about isothermal expansions in introductory chemistry. Finally, the validity of the approximation is confirmed by the agreement found between the model and our experimental results.

APPENDIX B: MODELING AIR DRAG

The force of drag on an object immersed in a fluid arises as a result of two distinct processes: (1) skin friction arising from shear forces within the liquid (laminar flow) and (2) transfer of momentum from the object to the surrounding fluid in the form of eddy currents (turbulent flow). The first process yields a force that is linearly dependent on speed of the fluid, while the second yields a process which is dependent on the square of the speed of the fluid. A qualitative explanation for each of these terms is given below.[12]

Consider the laminar flow of a fluid along the surface of an object. The velocity profile of the fluid will appear qualitatively as in Fig. 6, with the fluid at rest next to the object

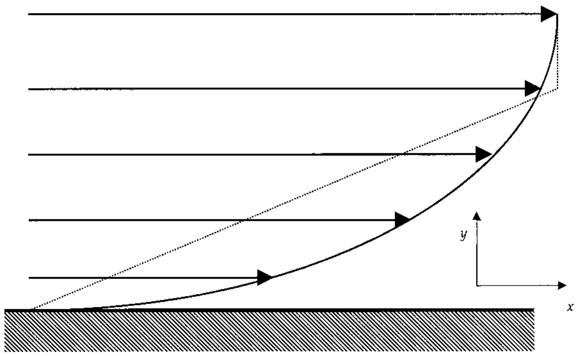

Fig. 6. Typical velocity profile of fluid in laminar flow in the vicinity of a solid object. The dotted line is the linear approximation used in text.

and flowing with a speed v at some distance away. This velocity profile results in a shear stress (force per unit area), σ, on the rocket given by $\sigma = \mu(dv/dy)$, where μ is the coefficient of viscosity, a property of the fluid. The velocity gradient can be approximated as Kv/w, where K is a dimensionless constant based on the shape of the object and w is a distance characteristic of the object. The viscous drag will be proportional to the product of the shear stress and the surface area of the object, or $D = \sigma A = \mu K A v/w$. Since K, A, and w are factors related to the geometry of the body, D can be rewritten as

$$D_1 = \mu k l v. \tag{B1}$$

Now k is a constant that depends on the shape of the object and l is a length (typically parallel to the flow). For example, a sphere has $k = 3\pi$ and l is the diameter of the sphere.

At higher speeds, the flow will not remain laminar, but will become turbulent. Then the object will impart momentum to the fluid by the generation of eddy currents and the drag will be based on an inertia force. The rate of momentum transfer is equal to the force exerted on the object, and can be determined in the following way. Consider the object moving a distance Δx through the fluid in a time Δt. The object will displace fluid with mass $m_{\text{fluid}} = k_1 \rho A \Delta x$, where k_1 is a geometrical constant, ρ is the fluid density, and A is the cross-sectional area. The average speed of the eddy currents will be proportional to the speed of the object (as long as the speed is not too high), so momentum imparted in the time Δt is $\Delta p = m_{\text{fluid}}(k_2 v_{\text{fluid}}) = k_1 k_2 \rho A \Delta x v$. Therefore, the rate of momentum transfer, and hence the drag, is (in the limit that Δt approaches zero)

$$D_2 = \Delta p/\Delta t = k_1 k_2 \rho A v (\Delta x/\Delta t) = \tfrac{1}{2} C_D \rho A v^2. \tag{B2}$$

Here, C_D is the familiar coefficient of drag.

While these are not rigorous derivations, they prove helpful in understanding the basic physics underlying the two forms of the velocity-dependent drag force. However, the student is still left with the question of which one to use. This question can be answered by considering the ratio of the two terms:

$$\frac{D_2}{D_1} = \frac{\frac{1}{2} C_D \rho A v^2}{k \mu l v} = \left(\frac{\frac{1}{2} C_D}{k}\right)\left(\frac{A}{l}\right)\left(\frac{\rho v}{\mu}\right). \tag{B3}$$

The first term in parentheses is a geometric factor that is typically on the order of one. The second term can be replaced with a linear dimension d (again, typically parallel to the flow). Now the ratio can be identified as

$$\frac{D_2}{D_1} \propto \frac{d \rho v}{\mu} = N_R, \tag{B4}$$

where N_R is known as the Reynolds number. For small values of the Reynolds number ($N_R < 1$), the flow is laminar and the drag is dominated by the viscous drag force, D_1. For large values of the Reynolds number ($N_R > 10\,000$), the flow is turbulent and the drag is dominated by the inertial drag force, D_2. In between, the drag can be modeled as the sum of the two forces. For this study, $\mu = 2 \times 10^{-4}\,\text{P}\,(2 \times 10^{-5}\,\text{kg/m s})$, $\rho = 1\,\text{kg/m}^3$, $d = 0.1\,\text{m}$, and $v = 20\,\text{m/s}$, so $N_R \approx 10^5$. Thus, we are justified using only the inertial drag force.

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[11]See, for example, Mary R. Boas, *Mathematical Methods in the Physical Sciences* (Wiley, New York, 1983), 2nd ed., pp. 558–562, or George Arfken, *Mathematical Methods for Physicists* (Academic, New York, 1985), 3rd ed., pp. 448–450, 513–593.
[12]See, for example, Ref. 8, Chaps. VIII and XII, or James W. Daily and Donald R. F. Harleman, *Fluid Dynamics* (Addison–Wesley, Reading, MA, 1966), Chaps. 8 and 9.