# § Eigencurve of Modular Symbols

C.M.: O.C. Modular Forms

Stevens: O.C. Modular Symbols

↳ Advantage: Includes theory of distributions <-> p-adic L-fct

—∘—

Let $p \nmid N$, $\Gamma = \Gamma_1(N) \cap \Gamma_0(p)$

$$\mathcal{H}_0 = \langle T_\ell \ (\ell \nmid pN),\ U_p,\ \langle a \rangle \rangle$$

Note: We don't include $U_\ell$ $(\ell \mid N)$

## §§ O.C. Mod. Symb over Affinoid

So far, we considered $L/\mathbb{Q}_p$ finite,

$$\kappa : \mathbb{Z}_p^\times \to L^\times,$$

$\rightsquigarrow \mathcal{A}_\kappa[r](L)$, $\mathcal{D}_\kappa[r](L)$, ...

This $\kappa \in \mathcal{W}(L)$ is a classical pt. We now extend to any affinoid

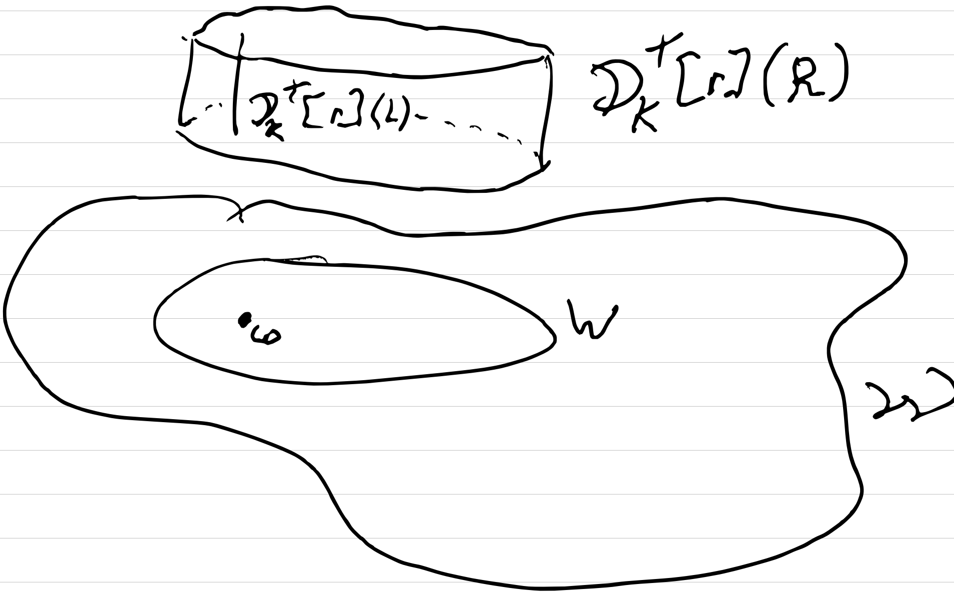
$$W = \mathrm{Sp}\, R \subset \mathcal{W} \iff K : \mathbb{Z}_p^\times \to R^\times$$

Given $\omega \in W(L) \subset \mathcal{W}(L) \iff \kappa : \mathbb{Z}_p^\times \xrightarrow{K} R^\times \xrightarrow{\omega} L^\times$


Lemma $\mathcal{D}^+[r](R) \otimes_{R,\omega} L \simeq \mathcal{D}^+[r](L)$

**Lemma** The action of $S_0(p)$ is compatible, i.e.

$$\mathcal{D}_K^+[r](R) \otimes_{R_\omega} L \cong \mathcal{D}_K^+[r](L)$$




**Lemma** Under (technical) hypothesis 7.1.3,

$$\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_K[r](R))$$

is potentially ON. (& all its submod. too)

From previous talk then, we know $U_p$ acts compactly on it $(0 < r < \min(r(K), p))$ so $(\quad)^{\leq \nu}$ makes sense (for $\nu$ adapted)

## § Compatibility / Restriction

<u>THM</u> $W' = Sp\, R' \subset W = Sp\, R \subset \mathcal{W}$, then for all $0 < r \leq r(\mathcal{W})$,

$$Symb_p(\mathcal{D}_K[r](R)) \hat{\otimes}_R R' \;\&\; S_p(\mathcal{D}_K[r](R'))$$

are linked.

(Namely, same adapted $\nu$'s & $(\;)^{\leq \nu} \cong$)

<u>Proof</u> Too long. (§7.1.2 of [B]) $\square$

## § Specialization

<u>THM</u> $W = Sp\, R \subset \mathcal{W}$, $L/\mathbb{Q}_p$, $\omega \in W(L)$

Then, $\exists$ natural $\mathcal{H}_0$-injection

$$S_p(\mathcal{D}^\dagger[r](R)) \otimes_{R,\omega} L \hookrightarrow S_p(\mathcal{D}^\dagger_\omega[r](\omega))$$

It is an $\cong$ if $\omega \neq 0$ & dim(coker) $= 1$ otherwise.

<u>Proof</u> Let $u$ = gen of $\ker(\omega)$ in $R$. Then,

$$0 \to \mathcal{D}^\dagger_K[r](R) \xrightarrow{\times u} \mathcal{D}^\dagger_K[r](R) \xrightarrow{\omega} \mathcal{D}^\dagger_\omega[r] \to 0$$

$\rightsquigarrow 0 \to H^1_c(\;) \xrightarrow{\times u} H^1_c(\;) \to H^1_c(\;) \to H^2_c(\;) \xrightarrow{\times u} H^2_c(\;)$

(with $H^1_c(\;) \cong S_p$ in the first three terms, and $H^2_c \cong H^0$)

By some lemma, $\ker = 0 \rightsquigarrow H^0 \xleftarrow{\times u} H^0$ if $\omega \neq 0$ & dim 1 o.w. $\square$

The image is denoted
$\mathrm{Symb}_p(\mathcal{D}_\omega^+[r](L))_{gl}$ ← "global" contribution

<u>Lemma</u> Indep of $W = \mathrm{Sp}\,R$ containing $\omega$.

<u>Rmk</u> Only interesting for $\omega = 0$.

## § Eisenstein Boundary Symbol

<u>Def</u> $E_2^{crit}$ is char $\mathcal{H}_0 \to L$ s.t.
$$T_\ell \mapsto \ell(\ell), \quad U_p \mapsto p, \quad \langle a \rangle \mapsto 1$$

<u>Lemma</u> $\exists\, \Phi_{E_2^{crit}} \in B\mathrm{Symb}_p(\mathcal{D}_0[r](\mathbb{Q}_p))$ for all $r > 0$ with $\mathcal{H}_0$-eigensys $E_2^{crit}$ and $\iota$-eigenvalue $-1$.

## § Construction

(ED1) $\mathcal{H}_0$

(ED2) $\mathcal{W}$ with admissible covering:

conn'd comp of $\mathcal{W}$ are all $\cong B(1,1)$

$\rightsquigarrow$ Let $\mathcal{C}$ = set of closed balls (in those open balls) w/ center $\in \mathbb{Z}$ and radius $\rho \in p^{\mathbb{Q}}$, $\rho < 1$.

(ED3) For $W = \mathrm{Sp}\, R$ in $\mathcal{C}$, choose any

$$0 < r < \min(r(W), \rho)$$

$$M_W := S_{\rho}^{\pm}(\mathcal{D}_K[r](B)),$$

where $\pm$ is w.r.t. involution $\iota = P(-1)^{\Gamma}$ (w/ natural $\psi_W : \mathcal{H}_0 \to \mathrm{End}_R(M_W)$)

<u>Rmk</u> Choice of $r$ doesn't matter b/c linked

The required cond's are:

(EC1) $U_p$ is compact on $M_W$

(EC2) $W' = \mathrm{Sp}\, R' \subset W = \mathrm{Sp}\, R \rightsquigarrow M_W \hat{\otimes}_R R'$ and $M_{W'}$ are linked

$\Rightarrow$ We get eigencurves $\mathcal{C}^{\pm}$ which are

- Equidim of dim 1
- Sep & rigid analytic sp / $\mathbb{Q}_p$

w/ $\kappa : \mathcal{C}^{\pm} \to \mathcal{W}$ & $T_\ell, U_p, \langle a \rangle \in \mathcal{O}(\mathcal{C}^{\pm})$

<u>Lemma</u> $T_\ell$, $U_p$ & $\langle a \rangle$ are power-bdd

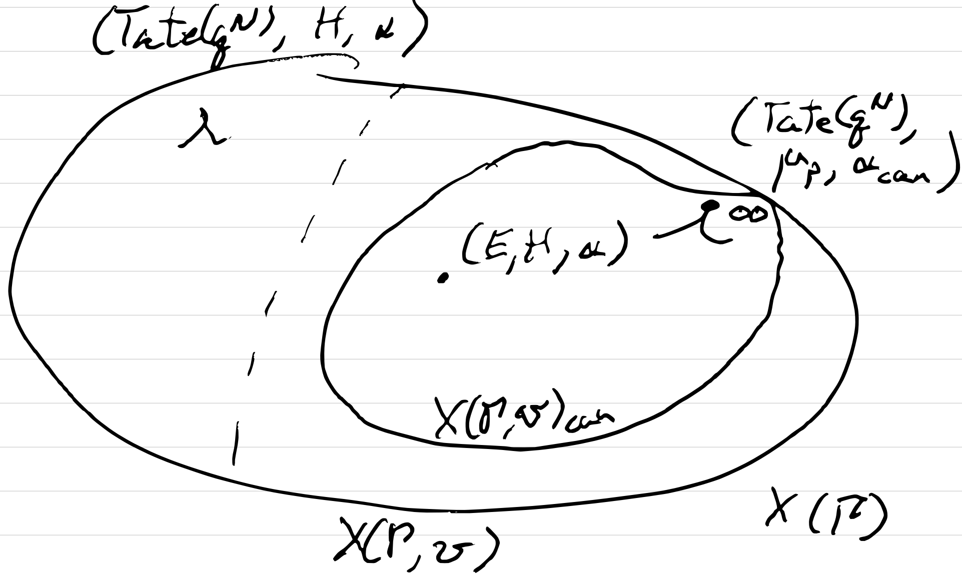
# § Comparison to CM Eigencurve

## §§ CM Construction

Same (ED1), (ED2), diff (ED3).

Let $\Gamma = \Gamma_1(N) \cap \Gamma_0(p)$ again.

Let $X(\Gamma)_{\mathbb{Q}_p}$ = rigid analytic mod. curve




Let $0 \le v < \frac{p}{p+1}$

- $X(\Gamma, v)(\mathbb{C}_p) = \{(E, H, \alpha) : E/\mathcal{O}_{\mathbb{C}_p}$ s.t. $|E_{p-1}(E, \eta)|_p \geq p^{-v}\}$

  $\cup$
- $X(\Gamma, v)_{can} = \{(E, \underset{\text{∃ by THM of Lubin}}{H_{can}}, \alpha)\}$

Def $R = \mathbb{Q}_p$ affinoid alg, then

$M^+[v](\Gamma, R) = R$-valued analytic functions on $X(\Gamma, v)_{can}$

This is pot. O.N. & commutes w/ B.C.

Fact $M_k(\Gamma, L) \hookrightarrow M^+[v](\Gamma, R)$

$f \longmapsto f/E_{p-1}^{k/p-1}$

THM (Coleman) Given $\kappa \in \mathcal{W}(R)$ and $v$ suff. small w.r.t. $(\kappa, R)$, $\exists$ weight-$\kappa$ action of $\mathcal{H}_0$ on $M^+[v](\Gamma, R)$

1) $U_p$ is cpct

2) action is compatible w/ BC

3) If $k \geq 2$, $\mathcal{H}_0$-equiv

$M_k(\Gamma, L) \hookrightarrow M^+[v](\Gamma, R)$

w/ wgt $(k-2)$ action

$\rightsquigarrow$ (ED 3): $W = \mathrm{Sp}\, R \subset \mathcal{W}$ in $C$, $\kappa: \mathbb{Z}_p \to R^\times$,

$M_W = M^+[v](\Gamma, R)$ (wgt $\kappa$)

$v$ small

$\rightsquigarrow$ eigencurve $\mathcal{C}$

The last statement means:

For $k \in \mathbb{Z} \subset W(\mathbb{Q}_p)$,

$$M^+[v](P,R) \otimes_{R,k} \mathbb{Q}_p = M^{\dagger}[r]_{k+2}(P,\mathbb{Q}_p)$$

(Restriction thm).

We also have Coleman's control thm

<u>THM</u> $M_{k+2}(P,\mathbb{Q}_p)^{<k+1} = M^+[r](P,\mathbb{Q}_p)^{<k+1}$

<u>Fact</u> there is a cuspidal eigencurve

$$S^+[v](P,R) \subset M^+[v](P,R)$$
$$\rightsquigarrow \mathcal{C}^0 \subset \mathcal{C}$$

Note: $f$ may be classical & not cuspidal but be in $S^+$

(E.g. $E_{2,crit} \in M(\Gamma_0(p))$

$= $ crit slope ref $E_2(z) - E_2(pz)$)

THM (i) $\exists!$ closed imm

$$\begin{tikzcd} \mathcal{E}^0 \arrow[r, hook] \arrow[dr] & \mathcal{E}^{\pm} \arrow[r, hook] \arrow[d] & \mathcal{E} \arrow[dl] \\ & \mathcal{W} & \end{tikzcd} \quad \begin{pmatrix} \text{all} \\ \text{red.} \end{pmatrix}$$

compatible w/

& $\mathcal{H}_0 \to \mathcal{O}(\mathcal{E}^0), \mathcal{O}(\mathcal{E}^{\pm}), \mathcal{O}(\mathcal{E})$

and $\mathcal{E} = \mathcal{E}^+ \cup \mathcal{E}^-$

(ii) For $k \in \mathcal{I} \subset \mathcal{W}(\mathbb{Q}_p)$, $L/\mathbb{Q}_p$,

$\exists$ injections of $\mathcal{H}_0$-mod's:

$$\begin{pmatrix} \nearrow \\ v=r=0 \end{pmatrix} S^{\pm}_{k+2}(\Gamma)(L)^{ss,\#} \subset \mathrm{Symb}_p^{\pm}(\mathcal{D}_k(L))^{ss,\#}_{gl} \subset M^{\pm}_{k+2}(\Gamma)^{ss,\#}$$

(iii) A sys. of $\mathcal{H}_0$-eigenvalues of f. sl. appears in $\mathrm{Symb}_p(\mathcal{D}_k)$

$\iff$ it appears in $M^{\pm}_{k+2}(\Gamma)$

(Except $\Gamma = \Gamma_0(p)$, $E_2^{crit}$ appears only in $\mathrm{Symb}_p(\mathcal{D}_0)$)

**Proof**

$$\begin{array}{rccc}
 & \mathcal{E}^0, & \mathcal{E}^\pm, & \mathcal{E} \\
 & \downarrow & \downarrow & \downarrow \\
x \in \mathcal{W}(L): & (M_x^0)^\#, & (M_x^\pm)^\#, & (M_x)^\# \\
 & \| & \| & \| \\
x = k \in \mathbb{Z}: & S_{k+2}^\dagger & \mathrm{Symb}_p^\pm(\mathcal{D}_k)_g & M_{k+2}^\dagger
\end{array}$$

We need classical structures:

(CSD1) $X = \mathbb{N} \subset \mathcal{W}$ (very $\mathbb{Z}$-dense)

(CSD2) $(M_k^0)^{cl} = S_{k+2}(\Gamma, \mathbb{Q}_p)^\#$

$(M_k^\pm)^{cl} = \mathrm{Symb}_\Gamma^\pm(V_k)^\#$

$(M_k)^{cl} = M_{k+2}(\Gamma, \mathbb{Q}_p)^\#$

(CSC) Want $\forall$ slope bound $\nu \in \mathbb{R}$, enough classical wgts $k \in \mathbb{Z}$ s.t.

$$(M_k^*)^{cl, \le \nu} \simeq (M_k^*)^{\le \nu}$$

(Works for $k+1 > \nu$ b/c of control thm)

By Chenevier's comparison thm:

$$(M_x^0)^{cl} \hookrightarrow (M_x^\pm)^{cl} \hookrightarrow (M_x)^{cl} \quad (H_p\text{-equiv})$$

$\rightsquigarrow \mathcal{E}^0 \hookrightarrow \mathcal{E}^\pm \hookrightarrow \mathcal{E}$

and $\mathcal{E} = \mathcal{E}^+ \cup \mathcal{E}^-$ b/c $M_{k+2} \hookrightarrow \mathrm{Symb}^+ \oplus S^-$ $\square$