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THE ARITHMETIC OF EVEN AND ODD NUMBERS: AN EXAMPLE OF MATHEMATICAL REASONING

0.1 HELLO

We hesitate to begin with a platitude, but let's – you and we – abandon any hesitations and embrace this thing wholeheartedly. In the spirit of earnest companionship and mutual trust, here's the platitude: the best and only fun way to learn mathematics is by doing mathematics. So let's do some math problems. Really. The sun is shining, and the wind is just right. Here we go!

First Activity

(0.1.1) Activity . a. Add two odd numbers together. Any two. Just pick your favorite two odd numbers and add them together.

b. All right, now, is the result even or odd? Try it again. Pick another pair of odd numbers (your, uhm, second favorite). Add them together and see whether the sum is even or odd.

c. Keep going. Five more times! (You didn't know this book was going to be so demanding. . . We pulled you in with all that talk about companionship and sun shining, then put you straight to work adding all our odd numbers together.)

d. Can you predict whether the result is even or odd when you add two odd numbers? And now – here's the amazing part, the point of entire book –can you explain *why* this happens?

In math when we observe a phenomenon that seems like it always happens, the first thing we do is try to describe it very precisely. Imagine a physician describing a really difficult case to a colleague. The description is probably going to be a bit more detailed than, say, "Hey buddy, this guy over here's got a stomach ache." It's probably going to have some technical vocabulary in it. When we're doing mathematics this precise description of something that always happens is called a *theorem*. Let's try it out!

(0.1.2) Theorem (*First Theorem*). The sum of two odd numbers is an even number.

It's concise. It seems correct. It says exactly what we want. It's perfect! (Well, except that it doesn't have a proof yet.) Notice how much information is packed into this little theorem. It tells us about adding *any* odd numbers! That's an infinite number of numbers!

Like with any technical description, you could spend a fair amount of time explaining Theorem 0.1.2 to someone who wasn't at all familiar with the terminology. Think about how you might introduce even and odd numbers to, say, a class of first graders. Think about how you might tell them about what a sum is.

We'll get to addition in Chapter 3 and talk more about even and odd in Chapter 5.

Let's go back to our two doctors for a moment. They're still there, huddled in the hallway, poring over their charts. When they're talking to each other, they're using the highly technical language of their profession. But when the doctor returns to her patient, she tries to explain the diagnosis in colloquial language. That's your job as a math teacher. You must understand the technical language of the subject in order to know what is going on well yourself. And you also must be able to translate it for your audience.

We will be developing language carefully in this book. We're not going to shy away from technical (and slightly terrifying) math words and symbols to serve precise, adult-level understanding. But we will also at times address how things might look and sound in the classroom.

Just a a doctors tries to find an underlying cause for symptoms they see, we try to give an explanation as to *why* our observations occur. This explanation always follows the statement of a theorem and goes by a lot of different names. Sometimes it's called your reasoning or rationale. It's your argument or justification. Most commonly, among mathematicians, it's called a proof. We will use all of these words for reasoning , but when the time comes to set the reasoning aside we'll call it a proof.

When you write down a theorem you're saying that something is true for everyone in the universe forever. We claim that no matter who adds two odd numbers, no matter when they do it, no matter which numbers they choose to add, the result will be even. That's pretty amazing! But if we're going to make such a grandiose claim, we'd better be sure.

The good news here is that there's no one correct way to write a proof. The bad news is that you probably have little if any experience in writing a proof, except perhaps in high-school geometry when they seemed like a strange exercise in writing in two columns. An analogy to keep in mind is in explaining to a kid who incessantly asks "why?" For an everyday example, suppose you told some young child that you were tired, and she asked why. You might say it was because you were coughing through the night, which interrupted your sleep. She could ask why again. You were coughing because you had gotten sick. You had gotten sick because you had gone out in cold weather without enough clothes on. That was because the weatherman didn't forecast a sudden change in weather, which in turn was because of some difficulties for certain storm fronts to be predicted. In mathematics, we usually put all of these "whys" in order: "Because it is difficult to predict certain storm fronts, the weatherman didn't forecast the the change in weather. That meant that even though I checked the weather, I went out without enough clothes." And so on.

Before we leave this example, observe that even at this level there could be more "whys" which come in between the whys given, such as "why is someone more likely to get sick when one is out in bad weather without enough clothing?" The appropriate level of detail, along with aspects such as amount of technical language, is an agreement between the person giving the proof and the person reading or hearing the proof. (Your instructors will be pretty demanding for your proofs, to prepare you for kids, who are the most demanding audience.)

Here's three different proofs for our theorem, all supplying an answer to "why?" As you're reading, try to pick out the core idea that's essential to each proof and think about how that idea could be scaled to different audiences. For each proof write down who you think the intended audience is. Make note of the technical vocabulary that's being used in each proof and how you might explain that vocabulary to the intended audience. While the proofs are given in adult-appropriate language, think about how you might translate these proofs into a age-appropriate activities for a class of second graders.

(0.1.3) PROOF (*of Theorem 0.1.2*). A number is odd if its rightmost digit is 1, 3, 5, 7, or 9. A number is even if its rightmost digit is 0, 2, 4, 6, or 8. To find the rightmost digit of the sum of two numbers, you only have to add the rightmost digits of the two numbers and take the rightmost digit of that. For example, consider the numbers 1345 and 629. The rightmost digits are 5 and 9. Adding these gives us 14, whose rightmost digit is 4. So, we expect the rightmost digit of $1345 + 629$ to be 4. And it is: $1345 + 629 = 1974$.

This tells us that in order to verify that the sum of *any two odd numbers* is an even number, we just have to check whether the sum of *any two odd digits* has an even digit on the right. It might not seem like a big deal at first, but this actually helps a lot! We've gone from talking about all the odd numbers (there are infinity of them) to talking about just five digits. We just checked this criterion for 5 and 9. We have to go through every case so that we're sure it always works:

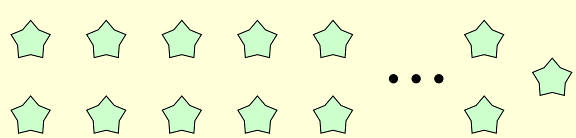
$$\begin{array}{cccc} 1+1=2 & 3+1=4 & \dots & 9+1=10 \\ 1+3=4 & 3+3=6 & \dots & 9+3=12 \\ 1+5=6 & 3+5=8 & \dots & 9+5=14 \\ 1+7=8 & 3+7=10 & \dots & 9+7=16 \\ 1+9=10 & 3+9=12 & \dots & 9+9=18 \end{array}$$

In every single one of these cases the rightmost digit is even.

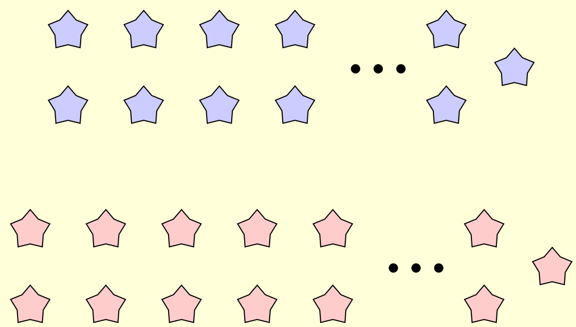
Follow the above chain of reasoning at least two times more to convince yourself that it gives a proof – a reason why for *any* two odd numbers their sum will be even. Proofs, like poetry, are meant to be read many times.

Here's a proof that gives a visual approach to Theorem 0.1.2.

(0.1.4) PROOF (*of Theorem 0.1.2*). One way to characterize odd numbers is: if you have an odd quantity of things then the things can be put in groups of two with one thing left over. For example, the things could be the students in your kindergarten class. If you have an odd number of students, whether it's 17, 21, or 49, when the students are paired there will be one left out. We can also represent this definition of odd numbers with a picture:



In the picture the things are the stars in the night sky, organized in groups of two, and as it happens there's one left out. We don't know exactly how many stars there are (and in fact we don't really want to know, because we want our proof to work regardless of how many there are, and because it would spoil some of the mystery of the cosmos) but we know it must be odd because there's one star by itself while the rest are paired. We indicate this uncertainty in the quantity by having dots represent some unknown amount of pairs of stars. Now suppose we have two collections of stars, which we can picture like this:



In this picture the blue stars are all the stars that we can see from the North Pole and the red stars are all the stars that we can see from the South Pole. We don't know exactly how many stars there are in either collection. Maybe there are 135668453433 blue stars and 7546453755421 red stars. It could really be anything. Well, almost anything – they must both be odd numbers because both collections come in pairs except for one star left over.

That was a way to describe what it means for a number to be odd . Similarly, even numbers count things which come in pairs with nothing left-out.

Given that language, the proof is simple. Suppose we add two odd collections together. We can pair those two left-out stars so that every star in the sum now appears with a partner. That means that the sum is even.

That's it! Once we think about odd and even in the right way, the proof is contained in the last paragraph. And this is enough to address what happens for *any* odd numbers.

In this second proof, notice how the dots mean something completely different than the first. In the first proof, we wrote the dots because we felt lazy, not because there was anything uncertain about how many cases there were to consider. In this proof, the dots mean that we don't really know how many stars there are, that it could just as well be *any* odd number.

OK, one more proof. This one is closer to something that a mathematician might write for another mathematician. It uses the language of mathematics, and in particular variables and algebra, but expresses exactly the same argument given in 0.1.4.

(0.1.5) Proof (*of Theorem 0.1.2*). A number is odd if it can be written as $2x + 1$, where x is some integer (The word *integer* means whole number, either positive, negative, or zero). A number is even if it can be written as $2x$, where x is some integer. To start, pick any two odd numbers. We can write them as $2n + 1$ and $2m + 1$. The sum of these two odd numbers is $(2n + 1) + (2m + 1)$. This can be simplified to $2n + 2m + 2$ and further simplified to $2(n + m + 1)$. The number $2(n + m + 1)$ is even because $n + m + 1$ is an integer. Therefore, the sum of the two odd numbers is even.

Whew! That was a lot of proving! It's mathematically sufficient just to prove a theorem once (if it's right, it's right) but in this book we're trying to do something more than just establish mathematical results. We're going to consider many ways of looking at certain basic math theorems, so that our understanding is flexible enough to support age-appropriate explanations for any classroom and to accommodate the variety of approaches kids will want to take.

This sort of understanding is part of mastery. *More on that in Chapter 2.*

Discussion

Now that we have some experience reading proofs, we can discuss mathematical reasoning and its role in education. Recall our example of a person explaining to a kid why they were tired. They provided a chain of reasoning, with each step following logically from the previous. Mathematical arguments are chains of reasoning, but applied to numbers, shapes, and data.

Each of the proofs just given starts with a definition and then proceeds until the desired conclusion is reached. When figuring out a mathematical proof we might start with the observed phenomenon and then search for evidence. But when writing a mathematical proof, we start with basic assumptions, called definitions and axioms, and reason forward.

One of the big questions to answer at this point is: Why does all this matter? Aren't we spending a lot of time on odd numbers when there are more serious topics to cover? (Like long division. Long division is as serious as it gets.)

There are a few good reasons for emphasizing genuine mathematical practice in elementary education. One is that it's an exercise in reasoning generally. It strengthens an aspect of cognition that is useful across many domains. Mathematics is a particularly great place to practice problem solving as there's nothing to "distract" from it, such as data collection in science activities or learning the trade of different crafts in art activities. And when kids really understand why we multiply multi digit numbers the way we do, they are doing what mathematicians do!

Secondly, learning is improved through structure. For example, memory recall of strings of letters or digits is greatly improved by breaking them down into pieces and then structuring those pieces in some artificial but helpful way. Indeed, there are a lot of clever tricks for remembering procedures and formulas in mathematics. You might still recall mnemonics such as FOIL and PEMDAS, as well as useful phrases such as "cross multiplication." But mathematics is more than a loosely connected collection of arbitrary facts. It has its own natural structure, and it's only through understanding these logical connections that the student can obtain real fluency, and possibly even enjoy the subject. And this sort of understanding leads to better recall of important facts, anyway.

An example of this second point is the multiplication table, a set of essential mathematical facts which we'll elaborate on in Chapter 4. Rather than just a collection of 144 independent pieces of information (1×1 through 12×12), it's helpful for the student to understand the multiplication table as a web of connected facts. For instance, the "six times row" is double the "three times row":

$$\begin{array}{cccccc} 3 \times 1 = 3 & 3 \times 2 = 6 & 3 \times 3 = 9 & 3 \times 4 = 12 & \dots & 3 \times 12 = 36 \\ 6 \times 1 = 6 & 6 \times 2 = 12 & 6 \times 3 = 18 & 6 \times 4 = 24 & \dots & 6 \times 12 = 72 \end{array}$$

This observation both reinforces recall, as for example to remember $6 \times 4 = 24$, the student can use the fact that 24 is twice as much as $12 = 3 \times 4$. But it also gives experience with what we will learn to call the associative law for multiplication. This law is part of a student's necessary foundational knowledge for learning algebra.

While there are certainly correct and incorrect mathematical statements, there is no exclusively right way to do *math. Like with all of the subjects you will be teaching, remember that context and audience inform the best practices for how should you present math.*

Turning back again to the three proofs of Theorem 0.1.2, we notice that these chains of reasoning have different characteristics. Proof 0.1.3 is more procedural, proof 0.1.4 is visual, and proof 0.1.5 is algebraic. Each starts with some definition for an odd number —either through the rightmost digit, through pairings with one element left over, or algebraically as a number of the form $2x + 1$. It's good to understand why these meanings are interchangeable, so that they are all are a valid place to start.

The amount of detail needed in a proof can depend upon the audience (contrast the crystalline brevity of Proof 0.1.5 with the organic abundance of Proof 0.1.4). In the exercises below, you'll find a question asking you to identify who the intended audience might be for each proof and describe how you would explain the proof's vocabulary to that audience.

Each of the proofs has some value. Proof 0.1.3 reinforces the importance and structure of the standard addition algorithm. Proof 0.1.4 is mostly pictorial, and could be appropriate for younger learners. In particular it could be shared before a student had learned about algebra or even multidigit addition. Proof 0.1.5 is an algebraic way to express 0.1.4. Compared to 0.1.4 it is more concise and and has a clarity that's characteristic of mathematical writing. But the language is very technical (there are variables!), and the exposition is terse.

Understanding multiple proofs in adult language helps prepare you for giving a variety of explanations in the classroom. Part of your work as a teacher will be to decide, based on the background and interest of your students, what reasoning is best to highlight in a given situation. More than that, you'll want to offer different types of arguments as a matter of course, so that your students can find among those what best supports their learning. As much as possible we will use multiple definitions to express the meaning of each concept that we study. We will look at different arguments to help you first reinforce your own understanding of the material and then begin the work of connecting your own advanced knowledge with age-appropriate explanations.

0.2 CONNECTIONS TO THE COMMON CORE

2.OA.3 Determine whether a group of objects (up to 20) has an odd or even number of members, e.g., by pairing objects or counting them by 2s.

Cluster heading for 2.OA.3-4 Work with equal groups to gain foundations for multiplication

In the Common Core, even and odd numbers are mentioned twice. Both times, they are used as interesting mathematics which can serve larger goals.

In second grade, students learn about even numbers as preparation for multiplication. Looking for groups of twos is the first example for looking at groups of equal size in general.

The very same odd plus odd is even and similar activities we share with you also occur in some second grade curricula. Of course, the proof we gave with variables would not be expected, but the pictorial proof could be developed (this pictorial proof has been the favorite mathematical argument of the last author's daughter, who is now nine years old). Establishing that that an odd number added to an odd number gives an even result would not seem to be "in" a very narrow reading of the Standards, but the reasoning about making an equal group (of two) certainly reinforces the main notion. Indeed, students should be doing mathematical work as they learn concepts, and the work of having to make an additional group of two is a good example.

Even and odd numbers are also mentioned in the cluster on generating and analyzing patters in fourth grade.

A good example is the task 3.OA Patterns in the Multiplication Table on Illustrative Mathematics (which you can find at *https://www.illustrativemathematics.org/illustrations/956*). Here we see that because multiples of odd numbers alternate between even and odd, while multiples of even numbers are always even, picturing them all together gives a sort of plaid pattern.

Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. For example, given the rule "Add 3" and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.

Patterns are not to just be noticed, but explained. While it is good to notice some regular behavior, it is much better to work to describe that behavior precisely and then understand and explain how that regular behavior is being produced, whether it be identifying mechanisms in science or finding proofs in mathematics. In this example, multiples of odd numbers alternate between even and odd because we repeatedly add to find the multiples, and odd plus odd is even while even plus odd is odd. These latter facts can be further explained, as we've done.

We've provided a first example around even and odd numbers also as an introduction to the Mathematical Practices. There are eight Mathematical Practices, which while only discussed briefly at the beginning the Common Core document are really the core of Common Core instruction.

1. Make sense of problems and persevere in solving them.

2. Reason abstractly and quantitatively.

3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

How these Practice Standards interface with the Content Standards is an interesting subject, worthy of an entire book to elaborate. We start understanding these by pointing out that each has key verbs which describe student actions. In our example of even and odd numbers, it is clear that that what we have on hand are some arguments or proofs. That already is novel, as traditionally it has been fine to develop K-12 mathematics without any rationale at all – odd plus odd is even because the teacher and textbook say so, and then maybe the examples we see agree with what they say.

But the third mathematical practice goes further, to aspire for students to be active in constructing viable arguments as well as critiquing the reasoning of others. How that can look in a classroom can vary widely, but the aims of the Practice Standards are not met by students only seeing a teacher make an argument, though that certainly can help clarify things at the end.

One of our main aims for these notes is to support your engagement in the Mathematical Practices. The most valuable tool for this are classroom and homework activities which we have designed. These not included in the main text so that instructors can use them with some flexibility and modify them as desired. Our goal in these notes is to support your basic understanding well enough, in particular going through key examples carefully, so that you can build on that understanding in novel ways which extend you as a learner.

Our discussion of odd and even provides an example of both adult-level and kid-appropriate viable arguments. We will see many more of these. We also hope that you have opportunities to persevere, to be precise, and to use structure and tools as we go through the upcoming chapters together.

Conclusion

In this chapter we wrote and proved our first theorem. We've had nothing but good times so far! In the exercises below, you'll have the chance to discover some more theorems about even and odd numbers.

You're working with other students and your instructor for now, but we'll see each other again next chapter. It'll be fun. We're going to learn ancient number system, including the Egyptian with staffs and pyramids and other funky pictures. But we don't want to spoil it all now. You've got some exercises to do.

We know that mathematical proofs demand more from a reader than other types of text. Thanks for reading!

0.3 EXERCISES

0.3.1. For each of 0.1.3, 0.1.4, 0.1.5:

- A.) Who do you think is the intended audience?
- B.) Write down any important vocabulary terms and how you would define those terms for the intended audience.

0.3.2. In 0.1.3, how many cases do we consider for the odd digit additions from $1+1$ to $9+9$. How many are repeats?

0.3.3. Answer the following questions related to 0.1.5.

- A.) Pick three odd numbers and write them as $2x+1$ for some integer x.
- B.) Pick three even numbers and write them as $2x$ for some integer x.
- C.) Explain each step in the simplification from $(2n+1)+(2m+1)$ to $2(m+n+1)$.
- D.) Work through all the steps of the proof with the numbers 37 and 59.

0.3.4. How would you explain Theorem 0.1.2 to a kindergarten class? Design an explanation or activity.

0.3.5. Work through the following activity.

> (0.3.1) ACTIVITY. Is the product of an even number and an odd number even or odd?

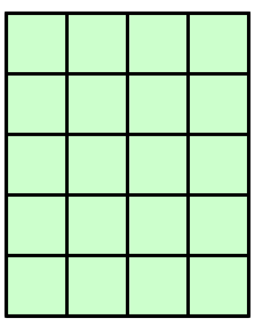
Recall that a *product* is the result (or, uhm, product) of multiplication. If you ever don't know what a word means, don't feel bad about having to look it up. Go through all the steps that we did in this chapter: perform experiments and gather data (that is, try it out with your favorite numbers), formulate a theorem, then prove the theorem from multiple perspectives. Here's a framework to get you started.

- A.) Multiply an even number with an odd number. Is the result even or odd? Repeat with at least five more pairs of numbers.
- B.) Write a theorem.
- C.) Let's do the first proof together.

> (0.3.2) PROOF (*First Proof of Your Theorem*). Let's use one of our definitions of even and odd numbers from this chapter. An even number is one that is twice some smaller number. For example, 4 is even because 4 is twice 2. 58 is even because 58 is twice 29. Odd numbers aren't twice any smaller number. OK. We've got our definitions. Now we have to show that the result (your theorem) is a natural consequence of these definitions. A lot of the time the hardest part in this is just picking the most convenient definitions.
>
> So, an even number times an odd number. Well, the even number is twice some smaller number. Let's call this smaller number SMALL. We're saying that our even number is twice SMALL. Good. Naming things is half the battle here. (That's all variables are: names for things.) Therefore, the product of the even number and the

odd number is twice the product of SMALL and the odd number. For example, if our even number is 8 and our odd number is 39, then SMALL is 4, and 8×39 is twice SMALL $\times 39$. Make sense? Well, that's it! Just read those last few sentences again. The product of the even number and the odd number is twice something else. That's exactly our definition of an even number. So, the product of an even number and an odd number is even.

D.) Write another proof that mostly uses pictures. Remember that the product of two numbers can be visually interpreted as a rectangular area. For example, 4×5:



E.) Write another proof that uses the technical language of mathematics: variables, equality signs, and so forth. A really good proof here probably shouldn't be any longer than three (3!!) lines.

0.3.6. Look back at the proofs to the previous question. Nowhere did you specifically use the fact that the second number is odd. In fact, exactly the same proofs give the following more general theorem:

> (0.3.3) THEOREM. The product of an even number with any integer is even.

There isn't really anything to do in this exercise. It's more of a mental exercise. The theorem from the previous exercise is just a special case of this one because an odd number is a type of integer.

0.3.7. (Challenge) Give a picture-based proof that the product of two odd numbers is odd.

0.3.8. Summarize the results of this chapter by completing the following table:

	+	×
ODD & ODD	EVEN	
ODD & EVEN		EVEN
EVEN & EVEN		