mod-two

$\bigoplus_n H^*(BS_n)$ is the free primitively generated $\mathbb{N}$-component "divided powers Hopf ring $\gamma_i \in H^{2^i-1}(BS_{2^i})$"

<u>Hopf ring</u> ring object in coalg's
two products, one coproduct
- cup product - zero b/t components

$\Delta_{n,m}$ induced by $BS_n \times S_m \longrightarrow BS_{n+m}$

$$UConf_n(\mathbb{I}^\infty) \times UConf_m(\mathbb{I}^\infty) \to UConf_{n+m}(\mathbb{I}^\infty)$$

$$\begin{tikzcd} UConf_n(\mathbb{I}^d) \times UConf_m(\mathbb{I}^d) \arrow[r, hook, "i"] & Conf_{n,m}(\mathbb{I}^d) \arrow[d, "p"', "\text{covering map}"] \\ & UConf_{n+m}(\mathbb{I}^d) \end{tikzcd}$$

labelled w/ 0 or 1

$\odot = p^! \circ i^*$

"0 pts" × "1 pts"

$W \times V \longleftarrow$ "$W \times V$" - some pts

$\downarrow$

$W \times V$ - some pts $/ (S_{n+m}/S_n \times S_m)$

"condition on n pts"
eg 2 pts share 5 coords

H^*: Δ, $\odot$

Alternately $S_n \times S_m \to S_{n+m}$.

$H \leq G$ finite

$BH \to BG$ covering map
$\parallel$ $\parallel$
$EG/H \to EG/G$

$(\cdot, \Delta)$ - a bialg (easy)

$(\odot, \Delta)$ - a bialg (harder) special!

Distributivity

$$a \cdot (b \odot c) = \sum (a_i' \cdot b) \odot (a_i'' \cdot c)$$

$$\Delta a = \sum a_i' \otimes a_i''$$

if $\odot$ = "union of two distinct conditions"
Divided power ~ repeating of a condition.
for $\odot$

$\{ x \odot x = 2x_{[2]}$ $\longleftarrow$ $x_{[n]} \cdot x_{[m]} = \binom{n+m}{n} x_{[n+m]}$

For ease of notation, divided powers given in subscripts

Related example (after Zelevinsky ~'80)

$\bigoplus_n Rep_k(S_n)$ is a Hopf ring

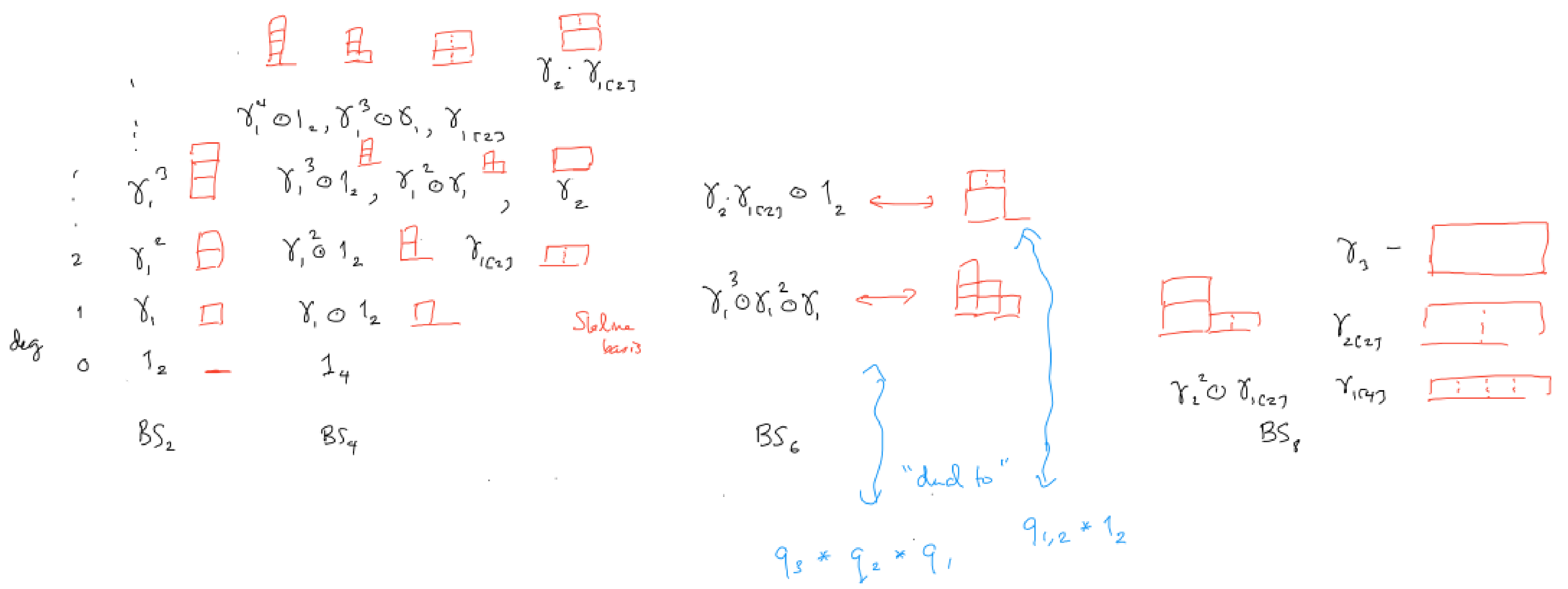
$\cdot = \otimes$ (internal to Rep S_n - zero between different S_n

$\Delta_{n,m}$ = restriction to $S_n \times S_m$

$\odot$ = induction product

$$V \odot W = Ind_{S_n \times S_m}^{S_{n+m}} V \otimes W$$

Hopf ring distributivity ⇒ Spanning set of the form
$m_1 \odot m_2 \odot - \odot m_k$.
each m_i monomial in · product



Cup products (on BS_4)

$\gamma_{1[2]} \cdot (\gamma_1 \odot 1_2)$

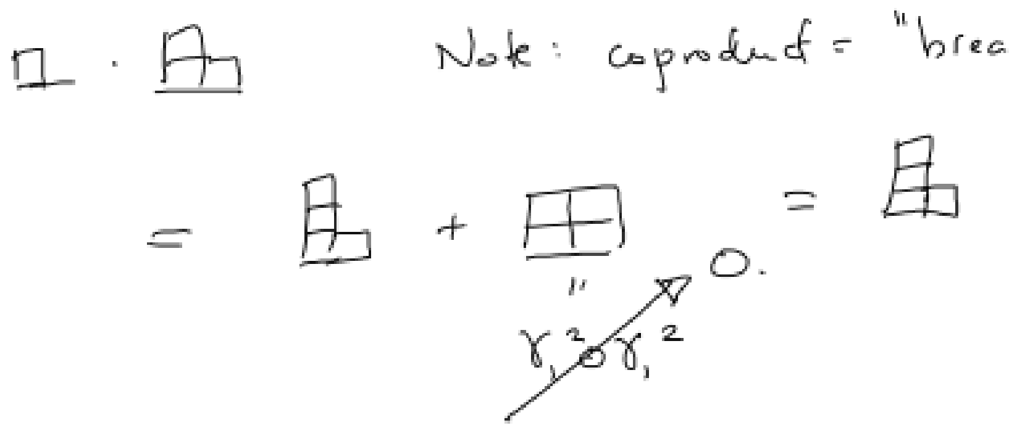
a b c

$\Delta \gamma_{1[2]} = \gamma_{1[2]} \otimes 1_0 + \underline{\gamma_1 \otimes \gamma_1} + 1_0 \otimes \gamma_{1[2]}$

$= (\cancel{\gamma_{1[2]} \cdot \gamma_1}) \odot (\cancel{1_0 \cdot 1_2}) + \gamma_1^2 \odot \gamma_1$

0 0

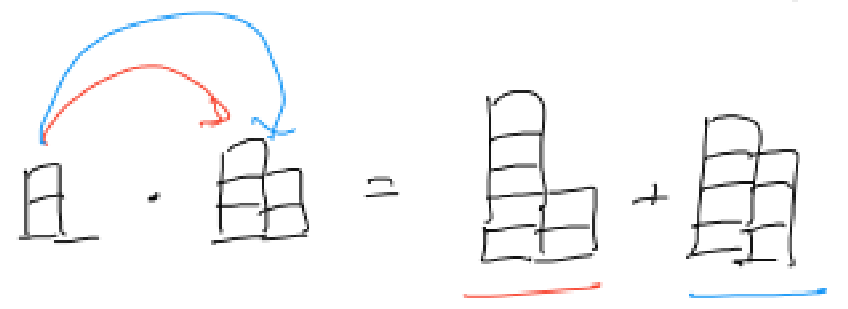
different components

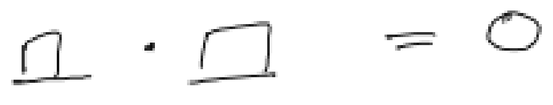


Note: coproduct = "break up along columns."

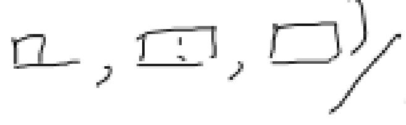
Distinction

divided power.

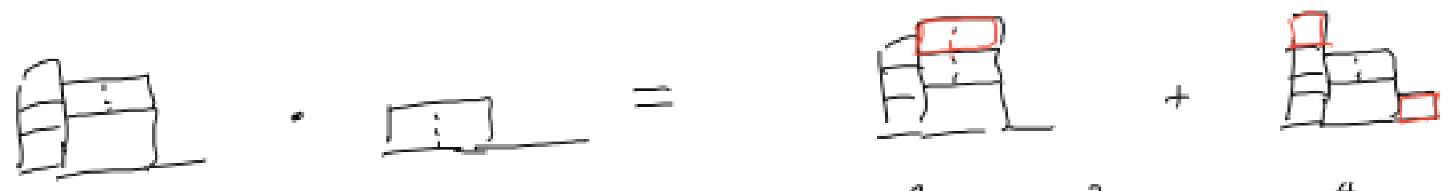




$\gamma_1 \odot 1_2 \cdot \gamma_2$ $\quad \Delta_{1,1} \gamma_2 = 0$
(primitively gen'd)

In fact $H^*(BS_4) \cong \mathbb{F}_2[$  $] /$  $= 0$.

EX
on BS_8



$\gamma_1^3 \odot \gamma_{1[2]} \gamma_2 \odot 1_2$ $\quad \gamma_{1[2]} \odot 1_4$ $\quad \gamma_1^3 \odot \gamma_2 \gamma_{1[2]}^2 \odot 1_2$ $\quad \gamma_1^4 \odot \gamma_2 \gamma_{1[2]} \odot \gamma_1$

<u>Propn</u>
Hopf ring monomials + distributivity
Skyline diagrams + "column matching algorithm"
give additive basis & mult rules for $\bigoplus_n H^*(BS_n)$.