Some proofs → Restriction to subgroups
→ Fox – Neuwirth cell structures / resolutions

$H \leq G$ Model $BH \longrightarrow BG$, $BH = EG/H$

In general finite covering maps

$Y \xrightarrow{\pi} X$, $\pi^!$ transfer

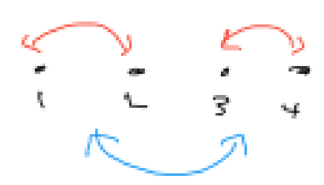
$C_n^{cw}(X) \xrightarrow{\pi^!} C_n(Y)$
$\sigma \longmapsto \Sigma \tilde{\sigma}$, $\pi\tilde{\sigma} = \sigma$

$\boxed{\pi_* \pi^! = \text{mult by } \# \text{ sheets}}$

$C^*(Y) \longrightarrow C^*(X)$
$\tau_w \xrightarrow{\pi_!} \tau_{\pi w}$.

<u>Cor.</u> $H \leq G$ p-sylow H_*, H^* mod p.
$H_*(BH) \twoheadrightarrow H_*(BG)$
$H^*(BH) \hookleftarrow H^*(BG)$

<u>EX</u> $G = S_4$. 2 sylow $= S_2 \wr S_2$



fib seq. $\mathbb{RP}^\infty \times \mathbb{RP}^\infty \longrightarrow BS_2 \wr S_2 \rightarrow \mathbb{RP}^\infty$
$BS_2 \wr S_2 = \underline{ES_2 \times_{S_2} (BS_2 \times BS_2)}$

Cell structure $C_*^{cw}(S^\infty) \otimes_{/S_2} (C_*(X) \otimes C_*(X)) \longrightarrow$ In general $\underline{ES_2 \times_{S_2} (X \times X)} = D_2 X$

Extended power.

<u>Prop</u> H_* / H^* of $D_2 X$ has additive basis

$H_* \begin{cases} 1 \otimes \alpha \otimes \beta \\ \sim 1 \otimes \beta \otimes \alpha. \end{cases}$ $\alpha * \beta$; α, β cycles. "$\alpha < \beta$"

$e_i \otimes \alpha \otimes \alpha$ ← $\phi_i \alpha$ – "formal" Kudo – Araki

Let $q_i \in H_i(\mathbb{RP}^\infty)$
So $H_*(S_2 \wr S_2) \twoheadrightarrow H(S_4)$
$q_i * q_j \longrightarrow q_i * q_j$
$\phi_i(q_j) \longrightarrow q_i(q_j) = q_{i,j}$

From this perspective main work in H_* is establishing Adem relations – 2 Sylow of BS_{2^n} is $S_2 \wr S_2 \wr \cdots$

First Adem relation: $q_2 q_0 = q_0 q_1$

In H^*

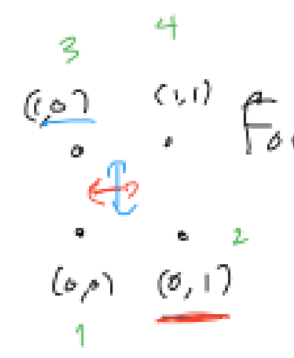
$V_1 \times V_1 := S_2 \times S_2 \hookrightarrow S_2 \wr S_2 \leq S_4$
$\left[(12), (34), (12)(34) \right.$

$V_2 := S_2 \times S_2 \xrightarrow{1 \times \Delta} S_2 \wr S_2$
$\left[(12)(34), (13)(24), (14)(23) \right.$

<u>Claim</u> S_4 maps injectively to direct sum of $H^*(V_1 \times V_1)$ $\oplus$ $H^*(V_2)$

After proving claim: $H^*(BG) \longrightarrow H^*(BH)$
$\searrow H^*(BH)^{WH} \subseteq H^*(BH)$. $WH = NH/H$.

$H^*(V_1 \times V_1) \cong \mathbb{F}_2[x_1, x_2]^W$ $W = S_2$
$=$ classical symmetric polynomials



For V_2, $V_2 \leq S_4$
$\mathbb{F}_2^2 \circlearrowleft_{+} \mathbb{F}_2^2 \rightsquigarrow \mathbb{F}^2 \hookrightarrow S_4$
$NH = Aff_2(\mathbb{F}_2)$ $W = GL_2(\mathbb{F}_2)$

$H^*(BS_4) \longrightarrow \mathbb{F}_2[x_1, x_2]^{GL_2(\mathbb{F}_2)}$. $\supseteq$ perm. matrices.

↑ fewer invts than symm polys.

<u>ex</u> $x_1 \longrightarrow x_1 + x_2$, $x_2 \longrightarrow x_2$; $x_1 + x_2 \longrightarrow x_1$

most "basic" invariant poly:
$d_3 := x_1 \cdot x_2 (x_1 + x_2) \in \deg 3$.
$d_2 := x_1^2 + x_1 x_2 + x_2^2 \in \deg 2$.

<u>Thm</u> (Dickson) $\mathbb{F}_2[x_1, x_2]^{GL_2 \mathbb{F}_2} \cong \mathbb{F}_2[d_2, d_3]$

<u>Moreover</u>

$V_1 \leq V_1 \times V_1 \leq S_4$, $V_2 \leq S_4$

$H^*(BS_4) \rightarrow \left(\lim\ H^*(BV_1 \times V_1) \rightarrow H^* BS \leftarrow H^*(BV_2) \right)$ $\cong$

First show injective.
Then will produce H^* to show $\cong$.
The Hopf ring presentation allow for generalization of this.