Claim

$S_4 \supseteq V_1 \times V_1$, $S_4 \supseteq V_2$

$H^*(BS_4) \to H^*(B(V_1\times V_1)) \oplus H^*(BV_2)$ injective

True in general that $H^*BS_n \hookrightarrow \bigoplus_{\substack{A \\ \text{elem ab}}} H^*BA$.

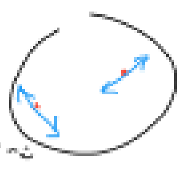
(Madsen-Milgram)

Quillen
$H^*(BG) \to \bigoplus_{\substack{A\\ \text{elab}}} H^*(BA)$ injective modulo nilpotents

$H^*(BS_2 \wr S_2) \to H^*B(V_1\times V_1)$

$q_i \times q_j$

basis: $x^i \odot x^j \leftrightarrow$ Thom class of (dual to)



on ℓ lin $\subseteq \mathbb{R}^{N-i}$

oth $\subseteq \mathbb{R}^{N-j}$

$y^i \cdot (x^j_{(2)}) \leftrightarrow$ Thom class of

$\therefore \subseteq \mathbb{R}P^{N-i}$

$\updownarrow$ each $\subseteq \mathbb{R}P^{N-j}$

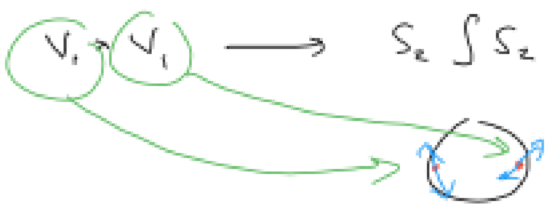
cup products $(x^i \odot x^j)\cdot(x^k \odot x^\ell) = x^{i+k}\odot x^{j+\ell} + x^{i+\ell}\odot x^{j+k}$

gen's $\left\{ \begin{array}{l} y^1 \cdot x^0_{(2)}, \quad y^0 \cdot x^1_{(2)} \\ y^i \odot x^j_{(2)} \cdot y^k \odot x^\ell_{(2)} = y^{i+k} \odot x^{j+\ell}_{(2)} \end{array}\right.$

$H^*(BV_1 \times V_1) \cong \mathbb{F}_2[z,w]$

$x^i \odot x^j \longrightarrow z^i w^j + z^j w^i$.

$V_1 \times V_1 \longrightarrow S_2 \wr S_2$

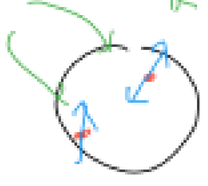


Restriction is thus injective on $x^i \odot x^j$ sub basis.

Key calculation

$\underbrace{S_2 \times S_2}_{1\times\Delta}^{V_2} \subseteq S_2 \wr S_2$

$x_{(2)} :=$ both in $\mathbb{R}^{N-1}$



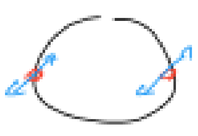
$? \in H^2 \longleftarrow x_{(2)} \in H^2(S_2 \wr S_2)$

$H_2(\mathbb{R}P^\infty \times \mathbb{R}P^\infty) = \mathrm{span}\begin{pmatrix} z_2 \\ z_1 \times w_1 \\ w_2 \end{pmatrix}$

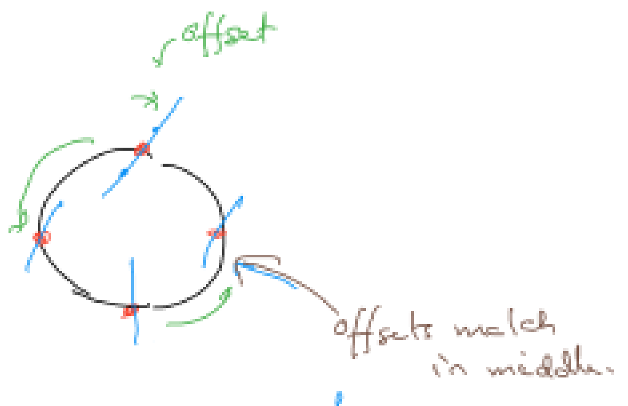
$x_{(2)}(z_2) = 0$

vary over S^2

$x_{(2)}(z_1 \times w_1) = 1$



vary over S^1



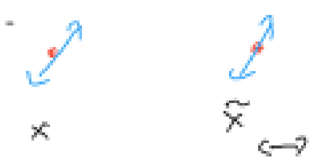
offset

map in by diagonal but not $\pitchfork$ so must deform diagonal

offsets match in middle.

instead of diagonal offset

$x_{(2)}(w_2) = 1.$



x $\quad \tilde{x} \leftrightarrow$ must be deformed

$T\mathbb{R}P^2 \downarrow \mathbb{R}P^2$

Such vanishes exactly once

$\Rightarrow \quad x_{(2)}{}^j \hookrightarrow H^*B(V_2)$

$\boxed{y^i \cdot x_{(2)}{}^j \hookrightarrow H^*(BV_2)}$

All together

H^*BS_4

$H^*BV_2 \qquad\qquad H^*BV_1 \times V_1$

0

d_2

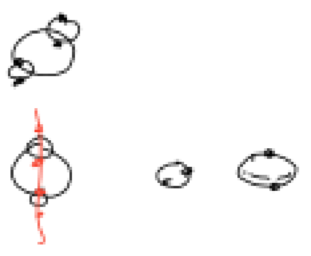
$d_3 = x_1 \cdot x_2 \cdot (x_1 + x_2)$

$y_1^i y_2^j + y_2^i y_1^j$

Recall $\square \Leftrightarrow$ 4 pts in sit inside $\mathbb{R}^{N-1}$

$\langle \square, q_{1,1} \rangle \neq 0$

$\langle \square, q_1 \times q_2 \rangle = 0$



$\square \mapsto d_3.$

b/c "nowhere else to go"

Some open (?) questions

1) H_*/H^* pairing for BS_n and full understanding of all maps on both sides

2) Derived divided powers operations on $H^*(D_n X)$ and $E^*(BS_n)$ (including $n=\infty$).

3) Yoneda Ext representatives for $H^*(BS_n)$.

4) Application to modular representation theory through support varieties.

5) Almost Hopf rings and functors with induction from $\{S_n\}$, $\{A_n\}$, $\{GL_n(\mathbb{F}_p)\}$ etc $\to$ rings.