Use config spaces to understand embedding spaces.

Need: Total fiber
$$\begin{tikzcd} X_\emptyset \arrow[r, "f"] \arrow[d, "g"] & X_1 \arrow[d, "h"] \\ X_2 \arrow[r, "\ell"] & X_{12} \end{tikzcd} \subseteq \mathrm{Map}(\mathbb{I}^2, X_{12}) \times \mathrm{Map}(\mathbb{I}, X_1) \times \mathrm{Map}(\mathbb{I}^0, X_\emptyset) \times \mathrm{Map}(\mathbb{I}, X_2)$$

$\gamma_1(0) = f(x)$

$\gamma_2(0) = g(x)$

s.t.

Another model:

Define $\mathrm{holim}\,(X_2 \xrightarrow{f} X_{12} \xleftarrow{g} X_1) \subseteq \mathrm{Map}(\mathbb{I}, X_{12}) \times \mathrm{Map}(\mathbb{I}^0, X_1) \times \mathrm{Map}(\mathbb{I}^0, X_2)$

$=$ homotopy pull back

$\gamma \quad x \quad y \quad z$

$X_\emptyset$ maps $\mathrm{holim}\,(X_2 \xrightarrow{f} X_{12} \xleftarrow{g} X_1)$

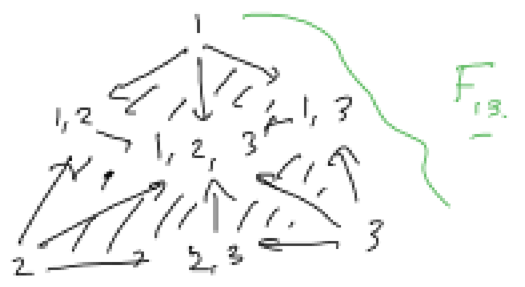
$x \longrightarrow$ const path @ img of x in X_{12}

Then Tot fib of square $\simeq$ hofib $X_\emptyset \to \mathrm{holim}\,(X_2 \to X_{12} \leftarrow X_1)$

So we can view $\mathrm{holim}\,(\quad)$ as an "approx" to $X_\emptyset$.

In general $2^n_{ne} \subseteq 2^n$, define $\mathrm{holim}\; 2^n_{ne} \to \mathrm{Top} \subseteq \prod_{F_S \subseteq \Delta^{n-1}} \mathrm{Map}(F_S, X_S)$

F_S? : $N\,2^n_{ne} \cong$ barycentric subdiv of $n-1$ simplex



Last time:

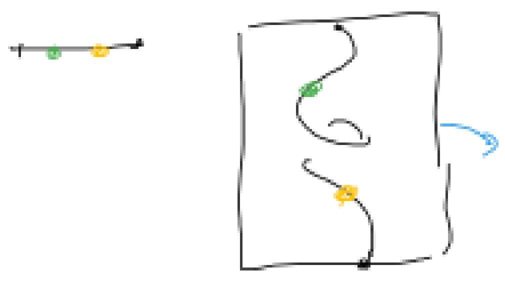
$$\mathrm{Emb}(\mathbb{I}, \mathbb{I}^d) = E_\emptyset \qquad E_S = \mathrm{Emb}(\mathbb{I} \setminus \bigcup_{i \in S} J_i, \mathbb{I}^d) \simeq \mathrm{Conf}_{\#S-1}(\mathbb{I}^d) \times (S^{d-1})^{d-1}$$

E

Showed that total fiber increases in connectivity when $d > 3$.

$\Rightarrow$ $E_\emptyset = \mathrm{Emb}(\mathbb{I}, \mathbb{I}^d) \longrightarrow \mathrm{holim}\,(\text{rest})$ is highly connected (as $\#S \to \infty$)

$n = 3$



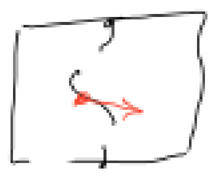
$\mathrm{Conf}_2\langle \mathbb{I} \rangle \cong \Delta^2$

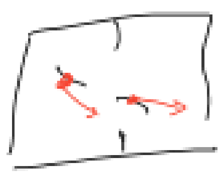
$\mathrm{Conf}_2\langle \mathbb{I} \rangle \longrightarrow \mathrm{Conf}_2\langle \mathbb{I}^d \rangle$

$\mathrm{Conf}_2\langle f \rangle$

$$\begin{tikzcd}[column sep=tiny] & & \mathrm{Emb}(\mathbb{I}, \mathbb{I}^d) \arrow[d] & \\ & & \mathrm{Emb}(\mathbb{I} \setminus J, \mathbb{I}^d) \arrow[dl] \arrow[dr] & \\ & \mathrm{Emb}(\mathbb{I} \setminus J_{13}, \mathbb{I}^d) & \mathrm{Emb}(\mathbb{I} \setminus J_{23}, \mathbb{I}^d) & \mathrm{Emb}(\mathbb{I} \setminus J_{12}, \mathbb{I}^d) \\ \mathrm{Emb}(\mathbb{I} \setminus J_3, \mathbb{I}^d) \arrow[r] \arrow[ur] & \mathrm{Emb}(\mathbb{I} \setminus J_{2,3}, \mathbb{I}^d) \arrow[u] & & \mathrm{Emb}(\mathbb{I} \setminus J_2, \mathbb{I}^d) \arrow[ll] \arrow[u] \end{tikzcd}$$

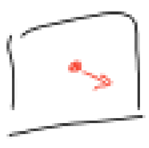
$f \mapsto \mathrm{Conf}_n\langle f \rangle$





"repeat" diagonal map

$$\begin{tikzcd}[column sep=small] & & \mathrm{Conf}_0'\langle \mathbb{I}^d \rangle \simeq * \arrow[dll] \arrow[d] & \\ \mathrm{Conf}_1'\langle \mathbb{I}^d \rangle \arrow[r] & \mathrm{Conf}_2'\langle \mathbb{I}^d \rangle & \mathrm{Conf}_1'\langle \mathbb{I}^d \rangle \arrow[l] & \\ \mathrm{Conf}_0'\langle \mathbb{I}^d \rangle \arrow[u] \arrow[r] & \mathrm{Conf}_1'\langle \mathbb{I}^d \rangle \arrow[u] & \mathrm{Conf}_0'\langle \mathbb{I}^d \rangle \arrow[l] \arrow[u] \end{tikzcd}$$



"punctured"

Thm The maps $\mathrm{Emb} \to (\text{left holim})$ & $\mathrm{Emb} \to$ right holim agree in homotopy category

"configs"

For $d > 3$ these induced maps on config spaces capture algebraic topology of embedding spaces.

Q $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^3) \longrightarrow \mathrm{holim} \subseteq \mathrm{Map}\,\Delta^n, (\mathrm{Conf}_n\langle \mathbb{I}^3 \rangle)$ st ...

$f \longrightarrow ev_n f = \mathrm{Conf}_n\langle f \rangle$.

What does this "see" on π_0?