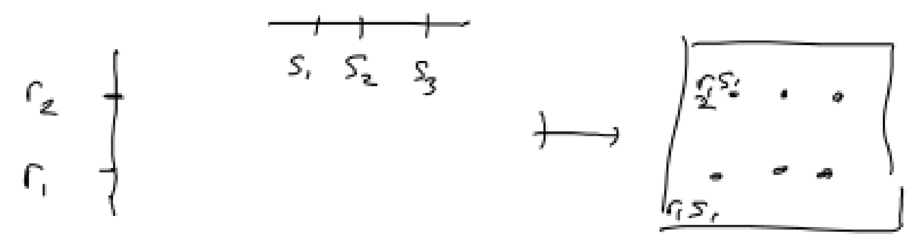
Focus instead on H_*. Use additional structure.
(I don't understand everything / as much as I'd like about "whole story together" – H_*, H^* & all structures)

- Each $B^n\mathbb{Z}/2$ is a top ab. gp $\Rightarrow$ H_* is a comm. Hopf algebra.

- For R a ring there are $B^nR \wedge B^mR \to B^{n+m}R$.



These endow $\bigoplus [X, B^nR]$ w/ structure of a ring.

$f: X \to B^nR$, $\quad g: X \to B^mR$

$X \xrightarrow{\Delta} X \times X \xrightarrow{f \wedge g} B^nR \wedge B^mR \to B^{n+m}R$

Propn coincides w/ cup product.

Propn $\bigoplus H_*(B^n\mathbb{Z}/2;\ \mathbb{Z}/2)$ is a Hopf ring

~~Bialg~~ Hopf monoid = monoid obj in coalg.
Hopf algebra = gp object in coalg's
Hopf ring = ring object in coalg's

Two products: $*_+$, $*_\times$ st ....

$$a *_\times (b *_+ c) = \sum_{\Delta a = \sum \alpha_i \otimes \alpha_i'} (\alpha_i *_\times b) *_+ (\alpha_i' *_\times c)$$

Thm (S. Wilson) $\bigoplus H_*(B^n\mathbb{Z}/2;\ \mathbb{Z}/2)$ is free Hopf ring over $H_*(B\mathbb{Z}/2;\ \mathbb{Z}/2)$

Start with $H_*(B\mathbb{Z}/2 = \mathbb{RP}^\infty)$

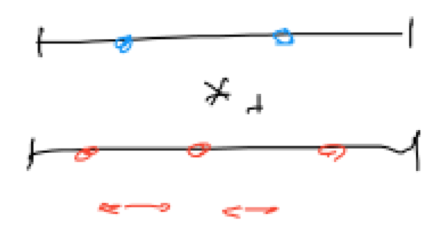
- dual to H^*. Polynomial gen'd deg 1.

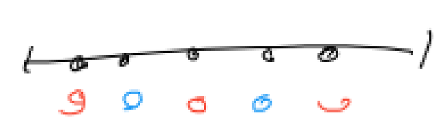
$\Rightarrow$ primitively gen'd $\Rightarrow$ H_* is divided powers (Milnor-Moore)

Geometrically $*_+$ multiplication is cellular

$y_i \in H_i(B\mathbb{Z}/2) \leftrightarrow$  i points.

EX $y_2 *_+ y_3$



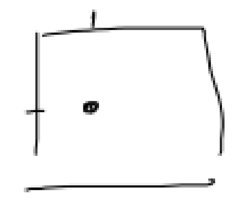
$= \binom{5}{2} \cdot$  y_5

$$y_i *_+ y_j = \binom{i+j}{i} y_{i+j}.$$

Divided powers.

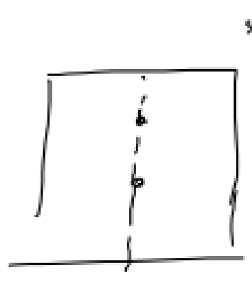
@ $\mathbb{Z}/2$. Exterior over y_{2^i}.

$y_3 = \binom{3}{1} y_2 *_+ y_1$

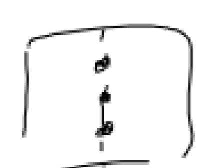
$n=2$ $\quad y_1 *_\times y_1$  $\xleftarrow{\text{dual to}}$ $z_2 \in H^2$ "fund. class"

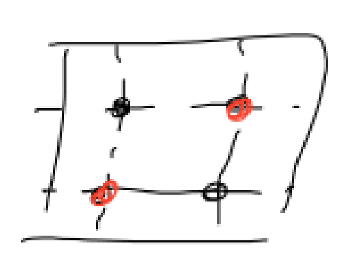
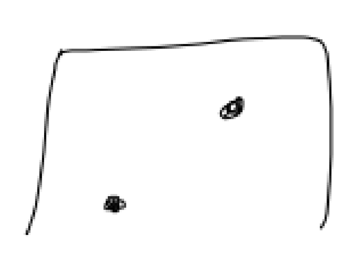
rep'd by $S^2 \to B^2\mathbb{Z}/2$
B/∂

$(y_1^{*_+ n} \longleftrightarrow z_n)$

$y_1 *_\times y_2$ 

$$y_1 *_\times y_3 = y_1 *_\times (y_1 *_+ y_2)$$
$$= \sum_{\Delta y_1 = y_1 \otimes 1 + 1 \otimes y_1} \cdots = (y_1 *_\times y_1) *_+ (1 *_\times y_2) + (1 *_\times y_1) *_+ (y_1 *_\times y_2) \quad ?$$

 $\simeq 0$. ?

$y_2 *_\times y_2$  $=$ 

relationship to (pairing w/) standard H^* statement?

More elementary / geometric proof using H_*, H^* together?