Central theorems in iterated loop space theory:

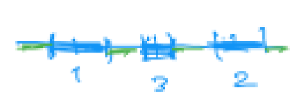
1. $\underline{Disks}^n$ operad acts on $\Omega^n X$.

2. $\Rightarrow H_*(\underline{Disks}^n)$ acts.
   ↖ non-equivariant & equivariant.

3. If $\underline{Disks}^n$ acts on Y & $\pi_0 Y$ gp $\Rightarrow Y \simeq \Omega^n X$ for some X.

4. $\Omega^n \Sigma^n Z \simeq (\text{free } \underline{Disks}^n \text{ space on } Z)^+$
   ↑ gp. complete, ΩBM.

$n = 1$. $\underline{Disks}^1 \simeq A_\infty$

1.  $\times (\Omega X)^3 \longrightarrow - \gamma_1 - \gamma_3 - \gamma_2 -$

   $\gamma_1, \gamma_2, \gamma_3$; basept

2. $H_*(\Omega X)$ is an associative algebra.
   (no new "equivariant" action here $n=1$)

3. (Stasheff) $A_\infty \circlearrowright Y$, $\pi_0 Y$ gp $\Rightarrow Y \simeq \Omega X$ some X

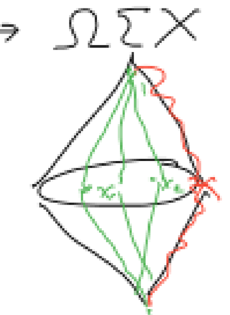
4. (James)

$$(X, *) \longrightarrow \Omega\Sigma X \quad \text{("inf. dim'l homogeneous")}$$
$$\searrow \quad \nearrow \simeq \text{(James)}$$
$$Mon(X, *) \quad \leftarrow \text{filtered by finite cx's}$$
$$\| $$
$$X \cup (X \times X) \cup (X^{\times 3}) \cup \cdots$$
$$(x_1, \ldots, x_i, *, x_{i+1}, \ldots x_k) \sim (x_1, \ldots x_i, x_{i+1}, \ldots x_k)$$
(basept.)

$$\Downarrow$$
$$H_*(\Omega\Sigma X) \simeq H_*(Mon(X); k) \cong \text{Free assoc alg. on } \bar{H}_*(X).$$
$$(= \text{Tensor algebra})$$

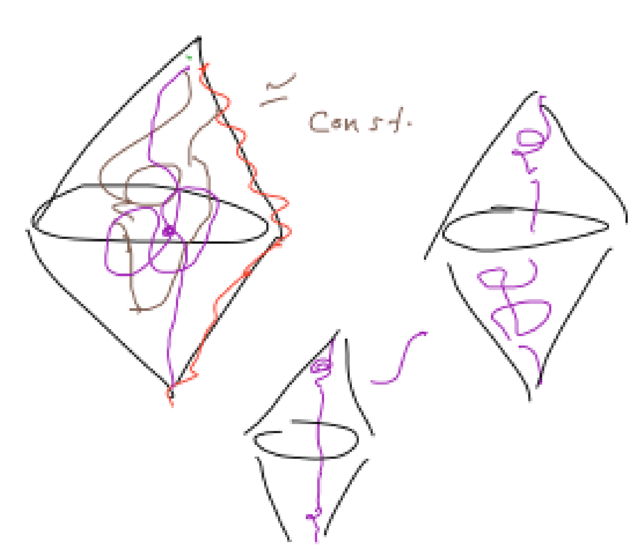
proofs calculate this & deduce thm.

What is map $Mon(X) \longrightarrow \Omega\Sigma X$?

$x_1, \ldots x_n \longmapsto$ 

Wishful proof of weak equivalence

$$S^n \longrightarrow \Omega\Sigma X$$
$$\searrow Mon(X) \nearrow ?$$



Definitions

- $Disks^d(n) = \{$ $\} \subseteq ((D^d) \times \mathbb{I})^n$
  ↑ centers ↑ radii.

- operads act to define alg structures ⟷ gps act to define symmetries

  proto example: $End(V)(n) = Hom(V^{\otimes n}, V)$ — proto example $Aut(X)$

  action $\mathcal{O} \longrightarrow End(V)$ — action on X: $G \longrightarrow Aut(X)$, $G \times X \longrightarrow X$.

  $$\mathcal{O}(n) \otimes V^{\otimes n} \longrightarrow V.$$

Ex $\mathcal{C} = Vect_k$. $\otimes = \otimes$.

Lie: $Lie(n) \otimes V^n \longrightarrow V$.

$[\,,[\,,\,]] \otimes v_1, v_2, v_3 \longrightarrow [[v_1, v_2], v_3]$

Let $Lie(n) = \langle [x_1, [x_2, x_3]], \ldots \rangle /$ Anti-symm. Jacobi
bracket expressions on $x_1, \ldots x_n$.

If Lie acts on V then use
$$Lie(2) \otimes V \otimes V \longrightarrow V$$
$$\cong k \qquad [\,,\,].$$

Check that further brackets are compatible

ex $Lie(2) \times Lie(2) \xrightarrow{\circ_2} Lie(3)$ ("insert")

$[x_1, x_2] \times [x_2, x_1] \longrightarrow [[x_2, x_3], x_1]$