Additional structure on $H_*(\Omega^d X)$ – Kudo-Araki operations

Pontryagin multiplication $\cup \Delta^i / \sim$ $\quad x \in \sigma_{x*}([M])$

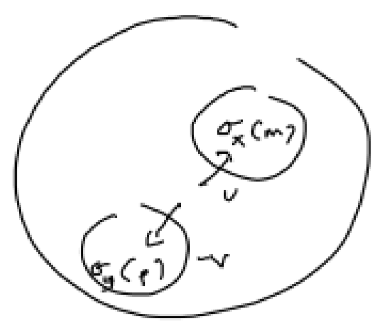
$x \in H_*(\Omega^d X)$, $y \in$

$$M^n \xrightarrow{\sigma_x} \Omega^d X \qquad P \xrightarrow{\sigma_y} \Omega^d Y \quad \rightsquigarrow \quad M \times P \to \Omega^d X \times \Omega^d X \xrightarrow{\mu_*} \Omega^d X$$

(composite: $\sigma_{x \cdot y}$)

Bracket "multiply in all directions"

$$S^{d-1} \times M \times P \longrightarrow \Omega^d X$$
$$v, \; m, \; p \longrightarrow \text{this}$$

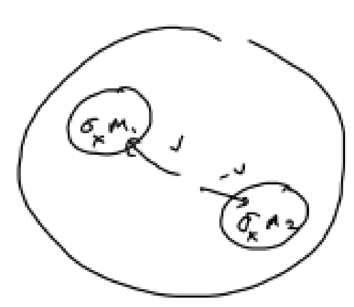


With $\mathbb{F}_2$-coefficients.

Kudo-Araki operations q_n $\quad n < d$

$M \to \Omega^d X \longmapsto$

$$\left(S^n \times_{S_2} (M \times M) \xrightarrow{\sigma_{q_n(x)}} \Omega^d(X) \right)$$
$$(v, \; m_1, \; m_2) / \sim$$



$$q_n : H_i(\Omega^d X) \longrightarrow H_{2i+n}(\Omega^d X). \qquad (\text{also called } Q^{i+n})$$

These satisfy Adem relations

$$q_m \circ q_n = \sum_i \binom{i-n-1}{2i-m-n} q_{m+2n-2i} \circ q_i$$

Thm (Cohen-Lada-May) $H_*(\Omega^d \Sigma^d X; \mathbb{F}_2)$

is a free (algebra + ...) object gen'd by $\overline{H}_*(X)$

where "object" means vector space w/

$*$ – product, antipode

$[\,,\,]$ – bracket deg $d-1$

Action of KA q_i subject to ...

Adem rels + compatibility.

"Switch" to H_* & H^* symmetric gps.

On the one hand this is purely algebraic:

$C_*^{cw} E_\Delta G$ – form a resolution of k over $k[G]$

$$\Rightarrow H^*(BG) \cong \mathrm{Ext}_{k[G]}(k, k)$$

<u>Lemma</u> $\quad \mathrm{Conf}_n(\mathbb{R}^\infty)$ is an ES_n

$S_n \circlearrowleft$ label permutation.

Pf $\quad \mathbb{R}^d \setminus \{(n-1) \text{ pts}\} \to \mathrm{Conf}_n(\mathbb{R}^d) \to \mathrm{Conf}_{n-1}(\mathbb{R}^d)$

$\quad \simeq \bigvee_{n-1} S^{d-1}$ $\quad$ So inductively, $\pi_* \mathrm{Conf}_n(\mathbb{R}^d)$ vanishes $* < d-1$

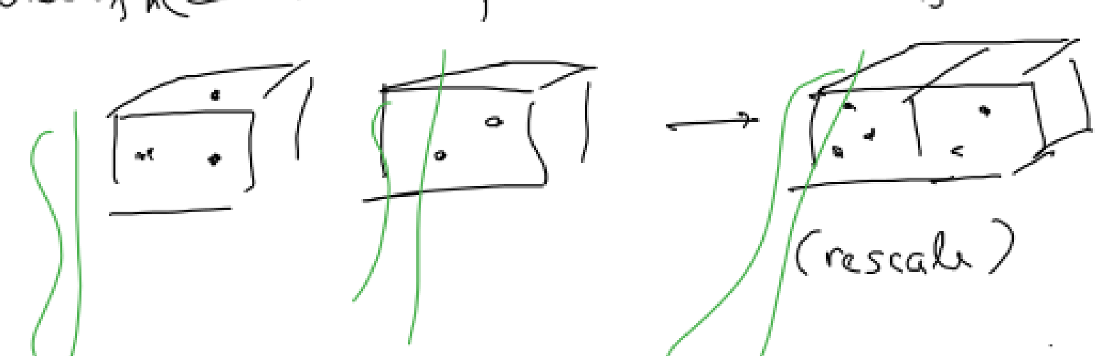
Take $\varinjlim \pi_* \mathrm{Conf}_n(\mathbb{R}^d) = 0$.

Let $U\mathrm{Conf}_n(X) = \mathrm{Conf}_n(X)/S_n$ $\qquad$ So $BS_n \simeq U\mathrm{Conf}_n(\mathbb{R}^\infty) \simeq U\mathrm{Conf}_n(\mathbb{I}^\infty)$

These have structure much as iterated loop spaces

$\coprod U\mathrm{Conf}_n(\mathbb{I}^\infty)$.

$$* : U\mathrm{Conf}_n(\mathbb{I}^\infty) \times U\mathrm{Conf}_m(\mathbb{I}^\infty) \to U\mathrm{Conf}_{n+m}$$



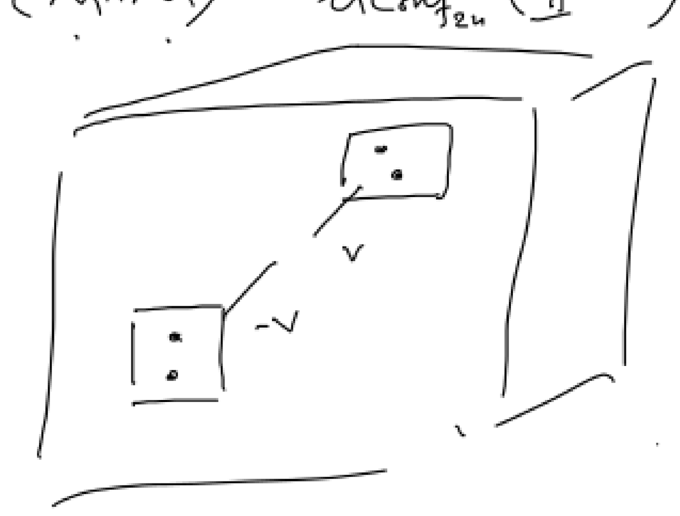
<u>Propn</u> htopic to

$$B_\Delta S_n \times B_\Delta S_m \longrightarrow B_\Delta S_{n+m}$$
$$B_\Delta(\text{incl } S_n \times S_m \to S_{n+m})$$

$$q_j : H_i(U\mathrm{Conf}_n(\mathbb{I}^\infty)) \to H_{2i+j}\, U\mathrm{Conf}_{2n}(\mathbb{I}^\infty)$$

$\mathbb{I}^\infty \;\cup\!/$

$$M \to U\mathrm{Conf}_n(\mathbb{I}^\infty) \longmapsto \left\{ S^j \times_{S_2} (M \times M) \to U\mathrm{Conf}_{2n}(\mathbb{I}^\infty) \right\}$$



This corresponds to a composite

$$H_i BS_n \to H_{2i+j} B(S_2 \wr S_n) \xrightarrow{B(\text{incl})} H_{2i+j} BS_{2n}$$