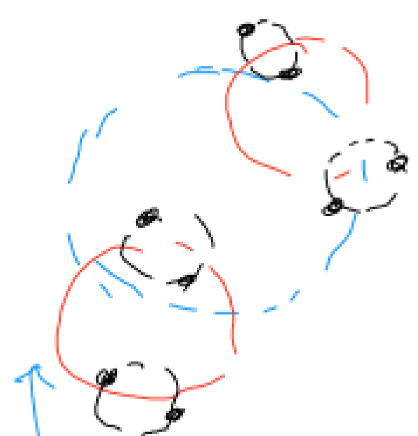
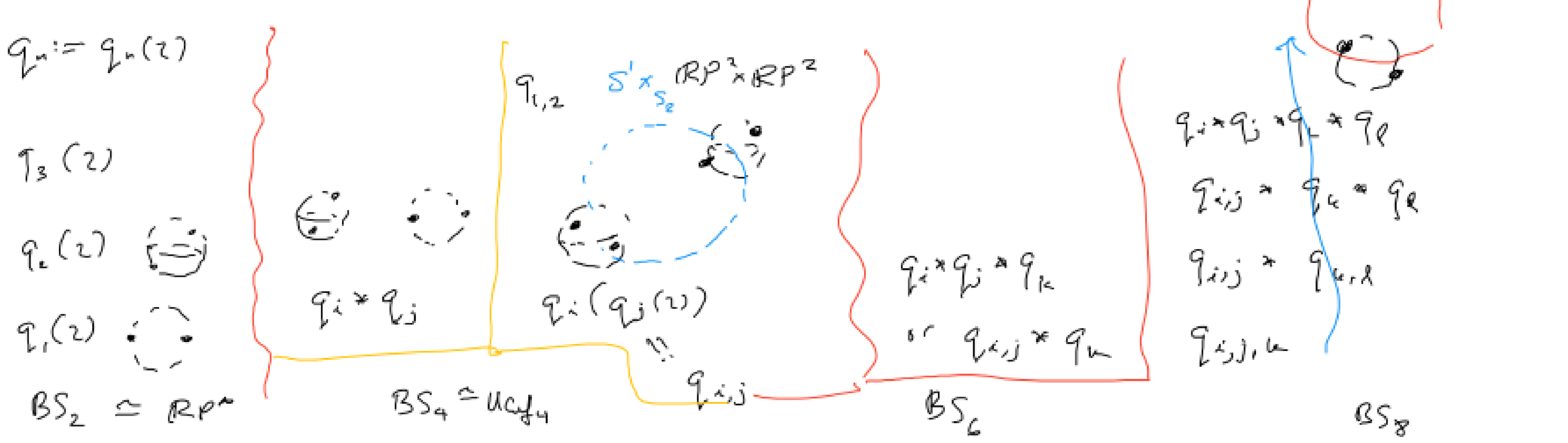
Thm (Cohen - Lada - May after Nakaoka)

$\bigoplus_n H_*(BS_n)$ is the free algebra on $*$ over q_j

gen'd by $\iota \in H_0(BS_1)$





So additively $\bigoplus_n H_*(BS_n; \mathbb{F}_2)$ known since Nakaoka 1960 (or 61)

H^* - - - - - -

"Bad news"
Cup product? $\quad \Delta q_I = \sum_{J+K=I} q_J \otimes q_K$ where J, K need not be admissible

Ex $\Delta q_{2,2} = q_{2,0} \otimes q_{0,2} + \text{reverse} + q_{1,1} \otimes q_{1,1}$

$\quad\quad = q_{0,1} \otimes q_{0,2} + \text{reverse} + q_{1,1} \otimes q_{1,1}$

Motivation: $\quad Conf_n \quad H_*$ – "orbital"
$\quad\quad\quad\quad\quad\quad\quad\quad H^*$ – "linear"

$\quad uConf_n \quad H_*$ – "orbital"
$\quad\quad\quad\quad\quad\quad\quad\quad H^*$ – "linear"

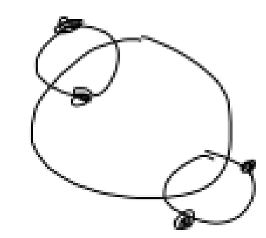
Ex BS_2

S^∞ detected by τ_w $\quad w = $ "2 pts share first i coords"

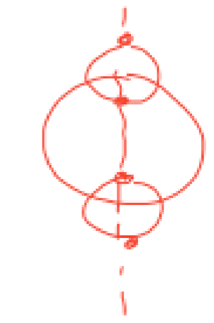
$q_1 * q_2$

$\longleftrightarrow$ 4pts $= \{$ 2 pts share 1 coord, 2 pts share 2 coords $\}$

$q_{1,1}$



$\longleftrightarrow$ 4pts share 1st coord



There is a product $\odot : H^*BS_n \otimes H^*BS_m \to H^*BS_{n+m}$

by "union of conditions." It is

tranfer for $BS_n \times BS_m \to BS_{n+m}$

Thm (Strickland - Turner) $\bigoplus H^*(BS_n)$ under
$\cdot$ = cup prod.
$\odot$ = transfer prod
Δ = coproduct (dual to $*$)

forms a Hopf ring.

Thm (Guerra - Salvatore - S-) $\odot$ has divided power operation "repeating condition".

Def $\gamma_i \in H^{2^i - 1}(BS_{2^i})$ be Thom class of "2^i pts share 1st coord"

Thm (Giusti - Salvatore - S-)

$\bigoplus H^*(BS_n)$ is free divided power W-component Hopf ring on classes $\gamma_i \in H^{2^i-1}(BS_{2^i})$

(and $1_n \in H^0(BS_n)$)