LINKING OF LETTERS AND THE LOWER CENTRAL SERIES OF FREE GROUPS

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Abstract. We develop invariants of the lower central series of free groups through linking of letters, showing they span the rational linear dual of the lower central series subquotients. We build on an approach to Lie coalgebras through operads, setting the stage for generalization to the lower central series Lie algebra of any group. Our approach yields a new co-basis for free Lie algebras. We compare with classical approaches of Magnus and Fox.

1. Introduction

Figure 1 below shows a classical picture of linking, along with an illustration of the type of linking we develop in this paper. In the former case, a one-manifold in S^3 is cobounded and intersected with another one-manifold. In the latter, a zero-manifold in S^1 is cobounded and intersected with another zero-manifold. This latter process is equivalent to choosing pairs of occurrences of a letter and its inverse, and counting instances of other letters in between – that is, linking of letters.

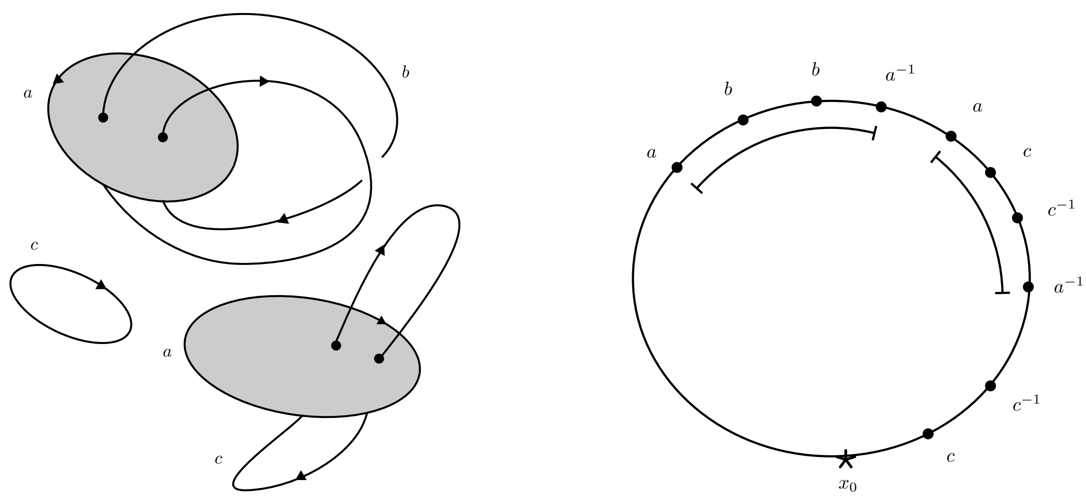

Figure 1. Classical linking is on the left; linking of letters on the right.

Linking, and its constituent processes of cobounding and intersecting, are staples in topology. Less familiar is that the process of cobounding and intersecting can be expanded and iterated to obtain higher linking numbers. Sinha and Walter show in [SW13] that such linking numbers detect all rational homotopy groups, spanning their linear duals. The analogous algebraic generalization is straightforward to conceptualize, at least in the free group setting. For example, the word $aba^{-1}b^{-1}$ is zero in the abelianization, and visibly is in the first commutator subgroup. It has linking number one, as there is a single b "caught between" an a-a^{-1} pair. We will see that this linking number obstructs its being in the second commutator

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subgroup. In a further example, all of the two-letter linking invariants of $[[a,b],c] = aba^{-1}b^{-1}cbab^{-1}a^{-1}c^{-1}$ vanish. But if we then consider occurrences of c between both an a-a^{-1} pair and a b-b^{-1} pair, that count is non-zero. We will see that this count, modeled on a way to distinguish maps from a four-sphere to a wedge sum of three two-spheres, obstructs the word being a three-fold commutator.

Our main results are to define purely algebraic linking and higher linking numbers between letters of words, and show that they perfectly reflect the lower central series filtration of free groups, spanning the linear duals of their subquotients. The subquotients of the lower central series for free groups constitute free Lie algebras, whose bases and linear duals have been well-studied [Reu93, SW11, MR96, BC06, Chi06]. The content of our work can be seen as lifting the definition of functionals from the free Lie algebra subquotients to the free groups themselves, as understanding the equivalence class in the subquotient is the goal rather than the starting point. Such functionals at the group level were given through the free differential calculus by Chen, Fox and Lyndon [Lyn58]. We compare our approach, showing that it is more efficient in examples, that it corresponds to a a new basis for the cofree Lie coalgebra, and that it is more closely related to the Quillen models for rational homotopy theory. We conjecture similar results for the linear dual to the lower central series Lie algebra of any finitely presented group.

Contents

2. Basic definitions

Our definitions are modeled on linking numbers of S^0 in S^1, with the S^0's corresponding to pairs of letters in a word and the cobounding intervals corresponding to sets of consecutive letters. The objects being linked and the cobounding objects can be defined either through subsets of a word or through functions on the letters of a word. Both descriptions are useful, more so together.

We let $w = x_1 \cdots x_k$ be a representative of a word in F_n, the free group on n generators (which we informally call letters), where each x_i is either a generator or its inverse. Thus, a word is an element of some cartesian power of set of generators and their inverses.

Definition 2.1. Let a be a generator, and x either a generator or the inverse of a generator. The intrinsic sign function is defined by

$$\operatorname{sign}^a(x) = \begin{cases} 1 & \text{if } x = a \\ -1 & \text{if } x = a^{-1} \\ 0 & \text{else} \end{cases}$$

Definition 2.2. Fix a generator a. A signed Linking Invariant Set for Tallying - in short, a signed list - is a set, possibly empty and possibly with repitition, of pairs

$$L = \{(\ell_1, \epsilon_1), \ldots, (\ell_p, \epsilon_p)\}$$

where each ℓ_i is some x_j which is equal to $a^{\pm 1}$ and each extrinsic sign ϵ_i is ± 1. We call such pairs signed letters. The total sign of such a pair, which by abuse we call $s(\ell_i)$, is $\epsilon_i \cdot \operatorname{sign}^a(\ell_i)$.

Given a list L the associated function f_L from the ordered set $(x_1, \ldots, x_k)$ to the integers sends x_i to the sum of the ϵ_i associated to its occurrences in L.

We say two lists are simply equivalent if their associated functions are equal.

Two lists are simply equivalent if and only they are related by a sequence of additions or removals of cancelling pairs $\{(a, 1), (a, -1)\}$.

We sometimes incorporate the generator which comprises a list in the name of the list, letting L_a and L'_a be lists of the letter a, et cetera. There is a standard list Λ_a of a generator a in w which is formed by having each $x_i = a^{\pm}$ in w appear in the list one and only one time, with $\epsilon_i = 1$. The associated function for this standard list is the indicator function for the subset of occurrences of a and a^{-1}.

As the acronym implies, we can tally or count a list.

Definition 2.3. The count ϕ of a list L_a is given by $\phi(L_a) = \sum s(\ell_i)$.

The counts of all generators in the standard list associated to a word in the free group determine its image in the abelianization. When these vanish, a word is in the commutator subgroup. We now define derived counts in the commutator subgroup. To do so, we first define cobounding.

Definition 2.4. An interval I in a word w is a nonempty set of consecutive letters. Such is determined by its first and last letters $\partial_0 I$ and $\partial_1 I$.

An oriented interval is an interval whose endpoints are signed letters with opposite total signs. We let $\partial I = \{(\partial_0 I, \sigma_0), (\partial_1 I, \sigma_1)\}$ where σ_0 and σ_1 are the extrinsic signs of $\partial_0 I$ and $\partial_1 I$ respectively. We let ϵ_0^I be the total sign of $\partial_0 I$, and similarly for ϵ_1^I. The orientation of I, denoted or I, is defined to be ϵ_0^I.

An oriented interval is defines a function f_I from the ordered set $(x_1, \ldots, x_k)$ to the integers whose value is 0 except on a consecutive set of letters whose endpoints have opposite total signs, and whose value on the set of consecutive letters is the total sign of the initial letter.

Geometrically, the orientation of an interval may be viewed as a total ordering which proceeds from the positively signed endpoint to the negatively signed endpoint.

Definition 2.5. Let $L = \{(\ell_1, \epsilon_1), \ldots, (\ell_p, \epsilon_p)\}$ be a nonempty list with $\phi(L) = 0$.

Define a cobounding $d^{-1}L$ as a set of oriented intervals $\{I_k\}$ such that each (ℓ_i, ϵ_i) occurs exactly once as either ∂_0 or ∂_1 of some I_k.

One can use the linear ordering on the letters of a word to define a canonical cobounding, but we will make use of the flexibility in such choices. Cobounding seems awkward to define through functions.

We now define linking of letters. Since linking is intersection with a choice of cobounding, the path forward is clear.

Definition 2.6. Let w be a word, L_a a list with $\phi(L_a) = 0$, and let $d^{-1}L_a = \{I_k\}$ be a choice of a cobounding. Let $L_b = \{(y_1, \epsilon_1), \ldots, (y_p, \epsilon_p)\}$ be a list with $b \neq a$.

Let (x_i, ϵ) be a signed generator and first define $(x_i, \epsilon) \cap I_k$ to be either $(x_i, \text{or } I_k \cdot \epsilon)$ if $x_i \in I_k$ or empty if $x_i \notin I_k$. Define the list $d^{-1}L_a \wedge L_b$, or equivalently $L_b \wedge d^{-1}L_a$, as the union of all $(y_i, \epsilon) \cap I_k$ as (y_i, ϵ) varies over L_b and I_k varies over $d^{-1}L_a$.

The simple equivalence class of the list $L_b \wedge d^{-1}L_a$, or $d^{-1}L_a \wedge L_b$, is that whose associated function is $\sum_{I \in d^{-1}L_a} f_{L_b} \cdot f_I$.

The list $d^{-1}L_a \wedge L_b$ witnesses the linking of the lists L_a and L_b. As $d^{-1}L_a \wedge L_b$ is itself a list, of generator b, we can iterate the process as long as the count of the lists which are produced vanish.

Definition 2.7. Recall $\Lambda_a(w)$ as the standard list defined by having each x_i which is equal to a or a^{-1} in w appear in the list one and only one time, with $\epsilon_i \equiv 1$. Define the depth of Λ_a to be zero.

Inductively, if L_a is a list of depth i of w with $\phi(L_a) = 0$ and L_b is of depth j with $b \neq a$ then we define the depth of a list $d^{-1}L_a \wedge L_b$ to be $i + j + 1$.

The (provisional) symbol of a depth-i list is its expression as an iterated application of d^{-1} and $\wedge$ to lists Λ_ℓ. We define the letter-linking function $\Phi_\sigma(w)$ to be $\phi(L)$, where L is a choice of list with symbol σ, when such a list exists.

See examples in Section A.1. The main result of this paper is that the letter linking functions Φ_σ determine the representative of a word in the lower central series Lie algebra of a free group. In the next sections we will develop relations between these functions, connect them to this lower central series filtration, and prove this main result. We will also further develop the combinatorics of such lists, and in particular introduce better notation. For example $\Phi_{d^{-1}(d^{-1}\Lambda_a \wedge \Lambda_b) \wedge \Lambda_a}$ is cumbersome, so we will say $\Phi_{((a)b)a}$ instead. But currently we prove the following, in steps.

Theorem 2.8. *The function Φ_σ is independent of choice of list with symbol σ and is independent of word representative of group element.*

We prove Theorem 2.8 by showing that the choices in cobounding and in representative of an element of the free group result in simply equivalent lists, which agree not only in their counts but in all of their "derived counts". This proof and others below rely on the geometry of intervals.

Definition 2.9. We say two intervals in a word are

- disjoint if they have no letters in common,
- contained if one is contained in the other,
- or otherwise we say they interleaved.

Proposition 2.10. *Let w be a word, and let L_a and L_b be lists of w with $\phi(L_a) = 0$. Then all choices of $d^{-1}L_a \wedge L_b$ are simply equivalent.*

Proof. Define an exchange of intervals to be replacing two intervals in a cobounding with two different intervals with the same four boundary points. Any two coboundings differ by a sequence of exchanges, so it suffices to analyze an exchange. We claim that in performing an exchange, intersecting with L_b yields lists which can only differ by a pair with opposite signs. Let v, x, y, z be the letters in the word w, in the order in which they occur, which are the endpoints of the exchanged intervals.

An exchange can occur between any two types of intervals. We focus on the case of exchanging between disjoint and interleaved. Similar arguments establish the other cases.

If there is an exchange between disjoint and interleaved coboundings of v, x, y and z then x and y must have the same total sign. Thus the two intervals in each matching have opposite orientations, and the leftmost intervals in the disjoint and interleaved coboundings have the same orientation, as do the rightmost.

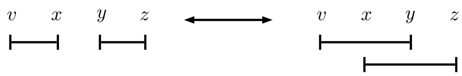


FIGURE 2. An exchange between disjoint and interleaved intervals.

In this case, occurrences of b between v and x and those between y and z are added once to the list $d^{-1}L_a \wedge L_b$ for both choices of cobounding, with the same signs. The occurrences of b between x and y do not get added at all for the disjoint cobounding, and in pairs with opposite signs for the interleaved cobounding. Thus the lists differ by pairs $(b, 1)$, $(b, -1)$, and so are simply equivalent. □

Corollary 2.11. *If L_a and L'_a are simply equivalent, as are L_b and L'_b, then so are $d^{-1}L_a \wedge L_b$ and $d^{-1}L'_a \wedge L'_b$, for any choice of coboundings.*

Proof. It suffices to consider L_a and L'_a which differ by one cancelling pair, say with L'_a having an additional pair. A choice $d^{-1}L_a$ of cobounding for L_a can be extended to one for $d^{-1}L'_a$, by adding the cancelling pair as an interval in the set. But no other letter including b can intersect this interval, so with these choices the resulting lists are the same. That $d^{-1}L_a \wedge L_b$ and $d^{-1}L_a \wedge L'_b$ differ by cancelling pairs when L_b and L'_b do is immediate. Applying Proposition 2.10, the resulting lists will be simply equivalent for any choices of coboundings. □

Proof of Theorem 2.8. That the function Φ_σ is independent of choice of list with symbol σ is immediate through inductive application of Corollary 2.11, which implies that all lists with the symbol σ will be simply equivalent and thus have the same count Φ_σ.

To show that the functions are well-defined on the free group, consider $w = w_1 w_2$ and $w' = w_1 a a^{-1} w_2$. We identify lists in w with lists in w' and inductively show that there are choices of depth-i lists on w' which differ by consecutive pairs – that is unions of the set $\{(a, \varepsilon), (a^{-1}, \varepsilon)\}$ where a and $a^{-}1$ are the added pair in w' – from the lists with the same symbol on w.

The base case of lists Λ_ℓ is immediate. Consider some $d^{-1}L_a \wedge L_b$. By inductive hypothesis, L_a on w' differs from the L_a with the same symbol on w by consecutive pairs. Choose a cobounding which starts by taking intervals whose endpoints are consecutive pairs before cobounding the rest of the list. Since no b's can be in the consecutive pair intervals, the lists $d^{-1}L_a \wedge L_b$ will be the same. Next for $d^{-1}L_b \wedge L_a$ the lists will differ by consecutive pairs, as intervals cannot have their endpoints between a and a^{-1}. Finally note that any $d^{-1}L_b \wedge L_c$ for b, c distinct from a will not differ between w and w'. □

3. Basic identities

3.1. **Symbols.** We first develop our notation for symbols, replacing our d^{-1} with parentheses, $\wedge$ with juxtoposition, and Λ_ℓ with ℓ.

Definition 3.1. A p-symbol is a parenthesized word in a generating set (no inverses) such that

- There is exactly one fewer pair of parentheses than letters.
- Every pair of parentheses contains exactly one letter which is not further parenthesized, which we call its free letter.
- Every pair of parentheses is either nested or disjoint.

The depth of a p-symbol is the number of pairs of parentheses (which is one less than the length of the word).

The first two conditions imply that at least one single letter is parenthesized by itself and one letter is unparenthesized. The third condition disambiguates repeated parentheses in the standard way. Examples include $a(b(c))$, $(a)b(c)$, and $(a(e))(a)(c)b$.

Definition 3.2. The shortened symbol of a depth-j list is obtained inductively as follows.

- The shortened symbol of Λ_a is a.
- If the shortened symbol of L_a is σ and that of L_b is τ then the shortened symbol of $d^{-1}L_a \wedge L_b$ is $(\sigma)\tau$ – that is, the shortened symbol of L_a parenthesized and followed by that of L_b.

From now on, we use shortened symbols to describe letter linking invariants.

Recall that in our definition of linking of lists, and thus the depth-i lists which define our linking invariants, the lists in question must be comprised of different letters. We capture this condition as follows.

Definition 3.3. Consider a pair of parentheses in a p-symbol, whose contents are of the form $(\sigma_1)\dots(\sigma_k)\ell$, where each σ_i is a symbol and ℓ is the free letter. A p-symbol is valid for this pair of parentheses if the free letters of σ_i are all different from ℓ. Define a symbol to be a p-symbol which is valid for all its pairs of parentheses.

We sometimes emphasize this distinctness condition by using the term "valid symbol."

3.2. **Homomorphism identities.** We now turn our attention to identities. In the previous section we showed that the letter linking functions Φ_σ are well defined, but we should recall that they are only defined on subsets of the free group, since the definition of $d^{-1}\Lambda_\mu \wedge \Lambda_\tau$ requires the vanishing of Φ_μ. Implicit in our statements of identities in this paper is that equalities hold only where all quantities involved are defined.

The Φ_σ on the identity element are all zero. More generally, if any $a^{\pm 1}$ does not appear in w but a does appear in σ then $\Phi_\sigma(w) = 0$. Next, we have algebraic compatibility.

Proposition 3.4. $\Phi_\sigma(w_1 \cdot w_2) = \Phi_\sigma(w_1) + \Phi_\sigma(w_2)$.

Proof. Inductively apply two facts. First, when all defined, the coboundings on $w_1 \cdot w_2$ can be chosen to be the union of those on w_1 and w_2. Secondly, $\wedge$ distributes over union of coboundings. □

Proposition 3.5. $\Phi_\sigma(w^{-1}) = -\Phi_\sigma(w)$.

Proof. Define compatible involutions on lists and their coboundings by taking inverses but leaving extrinsic signs unchanged. Inductively we can choose $\Lambda_\sigma(w^{-1})$ as the image of $\Lambda_\sigma(w)$ under this involution. Under this involution, counts of lists are multiplied by -1. □

The homomorphisms Φ_ℓ, for a generator ℓ, are the composite of the map from the free group to its abelianization followed by projection onto the ℓ-summand of the abelianization. We view the other homomorphisms Φ_σ as derived versions of abelianization.

3.3. **Leibniz identities.** The geometry of intervals gives rise to key relations.

Definition 3.6. The intersection $I \cap J$ of two oriented intervals is their intersection, with orientation given by the product of orientations.

The following two facts are immediate from the definitions.

Lemma 3.7 (Associativity)**.** *$(I \cap J) \cap K = I \cap (J \cap K)$ and $((x_i, \epsilon) \cap I) \cap J = (x_i, \epsilon) \cap (I \cap J)$.*

If S_i, for $i = 1, \dots, n$ are sets of intervals, define $\bigcap S_i$ to be $\bigcup_{I_1 \in S_1, \cdots, I_n \in S_n} I_1 \cap \cdots \cap I_n$.

Proposition 3.8 (Leibniz rule)**.** *$\partial(I \cap J) = (\partial I \cap J) \cup (I \cap \partial J)$. More generally*

$$\partial \bigcap_{i=1\cdots n} S_i = \bigcup_i S_1 \cap \cdots \cap S_{i-1} \cap \partial S_i \cap S_{i+1} \cap \cdots \cap S_n.$$

Here we are using $\cap$ from Definition 2.6 for intersecting sets of signed letters with lists.

Proposition 3.9. *[Leibniz Relation]*

$$\Phi_{\sigma_1(\sigma_2)\cdots(\sigma_{k-1})(\sigma_k)} + \Phi_{(\sigma_1)\sigma_2\cdots(\sigma_{k-1})(\sigma_k)} + \cdots + \Phi_{(\sigma_1)(\sigma_2)\cdots\sigma_{k-1}(\sigma_k)} + \Phi_{(\sigma_1)(\sigma_2)\cdots(\sigma_{k-1})\sigma_k} = 0.$$

The first two cases of this identity have distinct names. The $k = 2$ case, which can be rewritten as $\Phi_{(\sigma)\tau} = -\Phi_{\sigma(\tau)}$, is known as an anti-symmetry relation. It should be considered in contrast with the fact that by definition, or essentially by commutativity of intersection, $\Phi_{(\sigma)\tau} = \Phi_{\tau(\sigma)}$. We call the $k = 3$ case the Arnold identity, with the connection to the identity with the same name in topology, which figures prominently in our work in Section 4.

Proof of Proposition 3.9. The relation follows from a slightly more general fact. Let $a_1, \cdots, a_k$ be distinct letters, and $d^{-1}L_{a_i}$ coboundings of lists in those letters. Because $\partial \bigcap_i d^{-1}L_{a_i}$ is the boundary of a collection of intervals, its count is zero. Thus by the Leibniz rule

$$\phi(\bigcup_i (d^{-1}L_{a_1} \cap d^{-1}L_{a_2} \cap \cdots \cap \widehat{d^{-1}L_{a_i}} \cap \cdots \cap d^{-1}L_{a_k}) \wedge L_{a_i}) = 0.$$

Using this equality in setting L_{a_i} to be Λ_{σ_i} establishes the Leibniz Relation. □

3.4. **Commutator identities.** We start with an easier result which could independently be deduced from stronger results in the next section. Our current treatment helps illustrate both some identities we prove in the next section and proof technique for the stronger Theorem 3.12 below.

Proposition 3.10. $\Phi_{(\sigma)\tau}[v, w] = \Phi_\sigma(v)\Phi_\tau(w) - \Phi_\tau(v)\Phi_\sigma(w)$.

Proof. By convention $\Phi_\sigma(v)$ and $\Phi_\sigma(w)$ are defined, and thus so are $\Phi_\sigma(v^{-1})$ and $\Phi_\sigma(w^{-1})$. We use the the lists and cobounding intervals which define them to produce the list $\Lambda_\sigma(vwv^{-1}w^{-1})$ as the union of lists identified with $\Lambda_\sigma(v)$, $\Lambda_\sigma(w)$, $\Lambda_\sigma(v^{-1})$, and $\Lambda_\sigma(w^{-1})$, which we also assume to be chosen to respect the inverse involution. Call the generator in these lists a. The inverse involution matches occurrences of $a^{\pm 1}$ in $\Lambda_\sigma(v)$ with those of $a^{\mp 1}$ in $\Lambda_\sigma(v^{-1})$ and similarly for w, w^{-1}, through which we choose our cobounding $d^{-1}\Lambda_\sigma([v, w])$. Similarly choose $\Lambda_\tau([v, w])$ as the union of lists which can be identified with $\Lambda_\tau(v)$, $\Lambda_\tau(w)$, $\Lambda_\tau(v^{-1})$, and $\Lambda_\tau(w^{-1})$. See Figure 3 for a schematic.

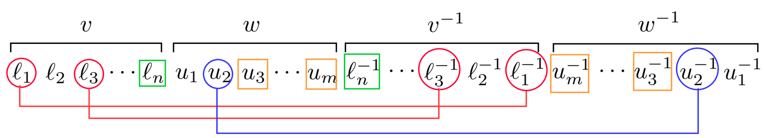


FIGURE 3. Schematic for Proposition 3.10.
$\Lambda_\sigma(v)$ & $\Lambda_\sigma(v^{-1})$ are denoted by red circles; $\Lambda_\sigma(w)$ & $\Lambda_\sigma(w^{-1})$ by blue circles; $\Lambda_\tau(v)$ & $\Lambda_\tau(v^{-1})$ by green squares; $\Lambda_\tau(w)$ & $\Lambda_\tau(w^{-1})$ by orange squares.

With these choices consider $d^{-1}\Lambda_\sigma([v, w]) \wedge \Lambda_\tau([v, w])$, starting with the intervals which cobound across v and v^{-1}. Such intervals do not intersect $\Lambda_\tau(w^{-1})$. Their intersections with $\Lambda_\tau(v)$ and $\Lambda_\tau(v^{-1})$ are matched under the inverse involution we have used to define our lists. There are full intersections with $\Lambda_\tau(w)$, meaning a total contribution of $\Phi_\sigma(v) \cdot \Phi_\tau(w)$.

Analysis of the coboundings across w and w^{-1} are similar, with only intersections with $\Lambda_\tau(v^{-1})$ contributing, yielding $-\Phi_\sigma(v)\Phi_\tau(w)$. □

Definition 3.11. Let G be a group. Inductively define the lower central series of groups by $\gamma_i G = [\gamma_{i-1}G, G]$, with $\gamma_0 G = G$. Inductively define the derived groups by $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$, with $G^{(0)} = G$.

We may inductively apply Proposition 3.10, starting with the immediate fact that Φ_ℓ vanishes on commutators, to see that Φ_σ vanishes on $F_n^{(i)}$ for i greater than the depth of σ. But we now show that the letter linking functions Φ_σ in fact vanish on the lower central series.

Theorem 3.12. *If the depth of a symbol is less than i, then the corresponding letter linking invariant vanishes on $\gamma_i F_n$. Thus if the depth of σ is equal to i, Φ_σ is defined on $\gamma_i F_n$.*

Proof. We argue inductively, starting with the immediate fact that Φ_ℓ vanishes on commutators.

Let σ be a symbol of depth $i-1$. Let $w \in \gamma_{i-1}F_n$ and $v \in F_n$, so $[w^{-1}, v]$ represents an element of $\gamma_i F_n$, and consider $\Phi_\sigma([w^{-1}, v]) = \Phi_\sigma(w^{-1}vwv^{-1})$. As Φ_σ is defined on w and w^{-1} we choose to use construct $\Lambda_\sigma[w^{-1}, v]$ starting with $\Lambda_\sigma w$ and $\Lambda_\sigma w^{-1}$, whose counts will cancel.

We complete $\Lambda_\sigma[w, v]$ through a construction of $\Lambda_\sigma vwv^{-1} \backslash \Lambda_\sigma w$, which allows us to show the count vanishes. Such a construction is the ultimate case of a second inductive claim that for any symbol τ of depth less than or equal to $i-1$, the list $\Lambda_\tau vwv^{-1} \backslash \Lambda_\tau w$ is defined and is comprised of letters with particular forms in $v \cup v^{-1}$ and w, namely:

- The letters in $v \cup v^{-1}$ are preserved under the v-v^{-1} involution.
- The letters in w are a union of lists $\Lambda_{\hat{\tau}_k} w$ for some collection of symbols $\{\hat{\tau}_k\}$ each of depth less than that of τ.

For brevity we call lists of letters in $v \cup v^{-1}$ and w with these properties type-one and type-two, respectively. Type-one pairs immediately have zero count, and type-two letters have zero count as well since each $\Lambda_{\hat{\tau}_k} w$ has zero count by our primary inductive hypothesis.

We prove this second claim through an induction on the depth of τ. We may use the vanishing statement of our theorem up to depth $i-2$. For depth zero, $\Lambda_\ell vwv^{-1} \backslash \Lambda_\ell w$ consists of occurrences of ℓ in vv^{-1}, which indeed occur in pairs preserved under involution, and thus is comprised entirely of type-one letters.

Next assume $\tau = (\mu_1)\mu_2$ where the claim has been verified for the μ_i. We choose the cobounding of the type-one subset of $\Lambda_{\mu_1} vwv^{-1} \backslash \Lambda_{\mu_1} w$ by cobounding pairs which correspond with one another under involution. We then cobound the type-two pairs, a cobounding which exists by inductive assumption because the depth of μ_1 is less than $i-1$. Consider the four cases for intersection arising in $(d^{-1}\Lambda_{\mu_1} vwv^{-1} \backslash d^{-1}\Lambda_{\mu_1} w) \wedge (\Lambda_{\mu_2} vwv^{-1} \backslash \Lambda_{\mu_2} w)$:

- The intersection of type-one pairs in $\Lambda_{\mu_2} vwv^{-1} \backslash \Lambda_{\mu_2} w$ with type-one cobounding intervals in $d^{-1}\Lambda_{\mu_1} vwv^{-1} \backslash d^{-1}\Lambda_{\mu_1} w^{-1}$ is a collection of type-one pairs.
- The intersection of a type-one cobounding interval for μ_1 with any list of letters in w, in particular any of the $\Lambda_{\hat{\mu_2}_k w}$, is the list itself. Thus the intersection of all such cobounding intervals with a union of $\Lambda_{\hat{\mu_2}_k w}$ is another such union. Because the depth of $\hat{\mu_2}_k$ is less than that of μ_2 it is less than that of τ.
- Type-two cobounding intervals are contained in w, so their intersection with type-one pairs is empty.
- The intersection of type-two cobounding intervals from some $d^{-1}\Lambda_{\hat{\mu_1}_j} w$ with all the type-two pairs from some $\Lambda_{\hat{\mu_2}_k} w$ is by definition $\Lambda_{(\hat{\mu_1}_j)\hat{\mu_2}_k} w$. Thus the union of all such intersections is the union of lists $\Lambda_{\hat{\tau}_\alpha} w$.

With this second induction step and thus the second induction claim established, we apply it for $\tau = \sigma$. We deduce that the count of $\Lambda_\sigma vwv^{-1} \backslash \Lambda_\sigma w$ is zero, completing our main induction. □

Corollary 3.13. *The Φ_σ of depth i are well defined on the lower central series subquotients $\gamma_i F_n / \gamma_{i+1} F_n$.*

To evaluate linking of letters invariants on the lower central series subquotients, and in particular show they span the linear dual, in the next section it is necessary to bring in the combinatorial approach to Lie coalgebras of [SW13].

4. Lie coalgebraic graphs and the main theorem

4.1. **Eil graphs and letter linking.** We connect with an approach to Lie coalgebras developed in [SW11] but with a slightly more combinatorial rather than algebraic starting point. We manipulate symbols, which are our parenthesized words which define letter linking homomorphisms, through using them as labels of graphs.

Definition 4.1. A symbol graph is an acyclic, connected, oriented graph whose vertices are labeled by symbols on a fixed generating set, so that if the vertices of two symbols are connected by an edge then their free letters must be distinct. Let $\mathcal{S}G$ denote the set of such and $\mathcal{S}G_{n,m}$ denote the subset with m edges and whose symbols have depths which sum to n.

The most important cases are $m = 0$, in which case we have a symbol labeling a solitary vertex, and $n = 0$ in which case we have an acyclic graph whose vertices are generators such that edges only connect distinct generators. We call the latter distinct-vertex Eil graphs because, as started in [SW11] and further developed and used below, they provide a model for the cofree Lie coalgebra on the generating set. The intermediate cases with both $n, m \neq 0$ are needed to relate symbols and Eil graphs.

Definition 4.2. Let v be a vertex in a symbol graph $G \in \mathcal{S}G_{n,m}$. When all resulting terms are valid symbols, the reduction of G at v, denoted $\rho_v G$ is the linear combination $\sum_{v \in \partial e} or_v(e) G_{v,e} \in \mathbb{Z}\mathcal{S}G_{n+1,m-1}$ where

- the sum is over edges e incident upon v,
- $or_v(e)$ is equal to 1 if e is oriented away from v and -1 if oriented towards v,
- $G_{v,e}$ is obtained from G by contracting the edge e and labeling its image in the quotient by $(\sigma)\tau$ where σ is the label of v and τ is the label of the other endpoint of e.

If any of the $G_{v,e}$ are not valid, we say the reduction of G at v is undefined.

For example if $G = {}_{a}\!\nearrow^{b}\!\searrow_{c}$ and v is the vertex labeled by b then $\rho_v(G) = \; {}_{-a(b)}\!\nearrow^{c} + {}_{a}\!\nearrow^{(b)c}$.

This definition is motivated by the definition of Hopf invariants in [SW13] through weight reduction in the Lie coalgebraic bar construction defined through graphs. If one follows the definition of weight reduction through for the fundamental group of a wedge of circles, one is led to reduction of symbol graphs.

Definition 4.3. If $w \neq v$ then by abuse we also use w to denote the corresponding vertex under identification in any $G_{v,e}$. By this convention, ρ_w is defined on all such $G_{v,e}$, and we let $\rho_w \circ \rho_v(G)$ be the composite defined by extending linearly, namely $\sum_{v \in \partial e} or_v(e) \rho_w G_{v,e}$, if all reductions are valid.

If $V = v_1, \ldots, v_k$ is a set of vertices of G let ρ_V be the composite $\rho_{v_k} \circ \rho_{v_{k-1}} \circ \cdots \circ \rho_{v_1}$, if defined. If this composite is not defined we say V is not valid.

The case of interest is when one reduces to a sum of graphs each of which has a single vertex decorated by a symbol, which we identify with the corresponding sum of symbols. For example $\rho_{b,a}$ of $G = {}_{a}\!\nearrow^{b}\!\searrow_{c}$ is $-(a(b))c + (a)(b)c$, while $\rho_{a,c}(G) = (a)b(c)$.

Definition 4.4. Suppressing the set of generators from notation, let $\mathcal{S}ymb_n$ denote the set of symbols of depth n, which is canonically identified with $\mathcal{S}G_{n,0}$. Extend the letter linking homomorphisms Φ_σ linearly to $\mathbb{Z}\mathcal{S}ymb_n$, with the domain of definition of a linear combination of homomorphisms given by intersection of the domains of the constituents.

We will find it fruitful to use not only symbols but also graphs to parametrize letter linking homomorphisms, as facilitated by the following main result.

Theorem 4.5. *Let $G \in \mathcal{S}G_{n,m}$. The letter-linking homomorphism $\Phi_{\rho_V G}$ is independent of choice of valid ordered set of $m-1$ vertices $V = v_1, \ldots, v_{m-1}$.*

In light of this theorem we shorten $\Phi_{\rho_V G}$ to just Φ_G.

In our example considering two different reductions of $G = {}_{a}\!\nearrow^{b}\!\searrow_{c}$ above, this theorem says that $-\Phi_{(a(b))c} + \Phi_{(a)(b)c} = \Phi_{(a)b(c)}$, which follows from the anti-symmetry and Arnold identities. Reduction of graphs thus gives a way of organizing relations between letter linking invariants.

The following combinatorics is at the heart of the proof of Theorem 4.5

Definition 4.6. Let S be a signed, partially ordered set with disjoint subsets A and B.

We say a sequence in S is

- alternating if consecutive terms have opposite signs,
- interleaving (of A and B) if it alternates between of two elements of A, followed by two elements of B, etc.,

We say a sequence crosses over some element c, which is not in A or B and is ordered with respect to all of their elements, whenever c occurs between consecutive terms in the sequence.

A crossing is homogeneous if the consecutive terms are both in A or both in B, or heterogeneous otherwise. The sign of such a crossing is the sign of the term in the sequence which is less than c in the total ordering (irregardless of whether that term was earlier or later in the sequence).

Lemma 4.7. *Let S, A and B be as above. For any alternating, interleaving sequence which begins and ends at the same point and any $c \notin A, B$ which is ordered with respect to them, the signed count of heterogenous crossings of c and that of homogeneous crossings are equal.*

Proof. Let $\{s_j\}$ be the sequence and suppose the sequence first crosses over c between s_i and s_{i+1} which are both in A or both in B, contributing $\epsilon = \pm 1$ to the count of homogeneous crossings. Let the next crossing over c be in k steps (that is, between s_{i+k} and s_{i+k+1}). Since the sequence is alternating and interleaving, consider k modulo four.

- If $k = 1$ or $3 \mod 4$ the next crossing contributes ϵ to the count of heterogeneous crossings.
- If $k = 0$ or $2 \mod 4$ the next crossing contributes $-\epsilon$ to the count of homogenous crossings.

In all cases the signed count of heterogenous crossings and that of homogeneous crossings are equal.

If the first crossing is heterogenous, the argument is the same, with the roles of heterogeneous and homogeneous interchanged. □

We apply this combinatorics to lists which arise in distinct reductions of symbol graphs. The general equivalence needed is the following.

Lemma 4.8. *Let L_a, L_b and L_c be lists of distinct letters a, b, c in a word w with $d^{-1}(L_a \wedge d^{-1}L_b)$ and $d^{-1}(d^{-1}L_a \wedge L_b)$ defined. Then the union of $L_c \wedge d^{-1}(L_a \wedge d^{-1}L_b)$ and $L_c \wedge d^{-1}L_a \wedge d^{-1}L_b$ with its orientations reversed is simply equivalent to $L_c \wedge d^{-1}(d^{-1}L_a \cap L_b)$.*

At the level of counts this follows from anti-symmetry and Arnold identities, but we need simple equivalence to have equality of further derived counts.

Proof. We fix choices of $d^{-1}L_a$ and $d^{-1}L_b$. A key first observation is that the boundaries of the intervals in $d^{-1}L_a \cap d^{-1}L_b$ and the boundaries of those in the union of $d^{-1}(L_a \wedge d^{-1}L_b)$ and $d^{-1}(d^{-1}L_a \wedge L_b)$ coincide.

We first quickly address of the case of containment of intervals in the L_a and L_b coboundings. If some $I \in d^{-1}L_a$ is contained in some $J \in d^{-1}L_b$ then I can be chosen in $d^{-1}(L_a \wedge d^{-1}L_b)$, and since $I \cap J = I$ it occurs in $d^{-1}L_a \cap d^{-1}L_b$. So any intersections of L_c with I would be added equally for the two sets named.

We thus focus on intersections of interleaving intervals from $d^{-1}L_a$ and $d^{-1}L_b$, which thus have one boundary point in L_a and one in L_b, of opposite total signs. With an eye to applying Lemma 4.7, set A to be the collection with multiplicity of all the elements of L_a which are boundaries of interleaving intervals from $d^{-1}L_a$ and $d^{-1}L_b$. Let B the elements of L_b which are such boundaries, and S be their union along with L_c. Order S using the ordering of letters of w.

By construction, points in A and B are all the boundaries of one interval from $d^{-1}L_a \cap d^{-1}L_b$. They are also the boundary of an interval $d^{-1}(L_a \wedge d^{-1}L_b)$ or $d^{-1}(d^{-1}L_a \wedge L_b)$ respectively. Thus the unions of these intervals form cycles (each point connected to two edges, each edge connected to two points). Starting with any point, following a cycle will define an alternating, interleaving sequence. The heterogeneous crossings of a point in L_c with this sequence are exactly contributions to $L_c \wedge d^{-1}L_a \cap d^{-1}L_b$ while the homogeneous crossings are contributions to $d^{-1}(L_a \wedge d^{-1}L_b)$ and $L_c \wedge d^{-1}(d^{-1}L_a \cap L_b)$. By Lemma 4.7 these are equal, from which we deduce the lemma. □

We can now prove that reduction of symbol graphs to symbols gives well-defined letter-linking invariants.

Proof of Theorem 4.5. Let $V = v_1, \ldots, v_{m-1}, v_m$ be a list of $m-1$ vertices at which a symbol graph in $\mathcal{SG}_{n,m}$ is to be reduced, followed by the remaining vertex v_m. Any two such lists differ by a sequence of transpositions, so it suffices to consider a V' which differs by a single transposition. Because the lists are the same up until the transposition, and thus will produce the same reductions up until that point, it suffices to consider a transposition of v_1 and v_2.

If v_1 and v_2 are not connected by an edge then the resulting reductions will be the same, so we assume there is an edge e between them, oriented say away from v_1 towards v_2. Let σ and τ be the symbols at v_1 and v_2 respectively. If this edge is the last one in the graph, then the resulting reductions are equivalent by anti-symmetry, so we consider the other cases when reduction occurs at both vertices. Each term in the linear combination of the reduction of G at v_1 and then v_2 correspond to a choice of edge incident to v_1 and an edge incident to v_2 in the quotient. If neither of these edges is e this term will be the same as the corresponding term in the reduction at v_2 and then v_1.

Thus we consider reduction at e along with a second edge f incident to v_1, say oriented away from v_1, connected to some vertex w labeled by symbol μ. There are two terms in the reduction at v_1 and then v_2 which correspond to contraction of e and f, namely e could come first and then f, resulting in the labeling symbol $\mu((\sigma)\tau)$ at the resulting vertex in the quotient, or f could come first and then e, resulting in $-\mu(\sigma)(\tau)$. There is one term in the reduction at v_2 and then v_1 as e must first be contracted then f, giving a labeling symbol $\mu(\sigma(\tau))$.

As the reduction of these terms will be identical after these contractions, it suffices to know for any w the union of the lists $\Lambda_{\mu((\sigma)\tau)}(w)$ and $\Lambda_{\mu(\sigma)(\tau)}(w)$ with its orientations reversed is simply equivalent to $\Lambda_{\mu(\sigma(\tau))}(w)$. But this is the content of Lemma 4.8, setting $L_c = \Lambda_\mu(w)$, $L_a = \Lambda_\sigma(w)$ and $L_b = \Lambda_\tau(w)$. □

To make full use of this reduction, we need the following simple piece of combinatorics.

Proposition 4.9. *For any symbol σ there is a graph G and sequence of vertices V such that $\rho_V G = \sigma$.*

Proof. One such graph G has a vertex of every pair of parentheses along with a vertex for the entire symbol. Each vertex is labeled by the free letter for the corresponding pair of parentheses, or respectively the free letter for the symbol. There is an edge from the vertex of a set of parentheses to the set of parentheses which immediately contains it, or respectively to the vertex for the entire symbol for the parentheses not contained in any others. By reducting at any list of vertices whose order extends the partial order by containment of parentheses, we obtain σ as the reduction. □

4.2. **The configuration pairing.** The lower central series filtration of a group is universal among filtrations whose subquotients form a Lie algebra. In the case of free groups, the resulting Lie algebra is free. In [SW11] the second author and Ben Walter developed an approach to free Lie algebras and their linear duals, starting with an operadic perspective.

Definition 4.10. Fix a set $x_1, \ldots, x_n$ of generators of F_n. Let $UC(n)$ denote the set of commutators in which each generator occurs a unique time, and let $\mathcal{L}ie(n)$ denote the submodule of $\gamma_n F_n / \gamma_{n+1} F_n$ generated by $UC(n)$.

Combinatorially, these can be represented by trees.

Definition 4.11. Let $\mathcal{T}r(n)$ denote the set of isotopy classes of half-planar, uni-trivalent trees with leaves labeled by integers $1 \ldots n$.

Represent a commutator $w \in UC(n)$ by an element $T_w \in \mathcal{T}r(n)$, starting with a one-edge tree with leaf label i as T_{x_i}. Then define $T_{[w,v]}$ to the tree formed by taking T_w and T_v and grafting them to a single (rooted) trivalent vertex, with T_w on the left.

Then $\mathcal{L}ie(n)$ is isomorphic to the quotient of $\mathbb{Z}\mathcal{T}r(n)$ by linear combinations corresponding to anti-symmetry and Jacobi identities.

Definition 4.12. Let $\mathcal{G}r(n)$ denote the subset of $\mathcal{S}G_{0,n}$ given by acyclic oriented graphs in which each generator occurs exactly once.

Thus $\mathcal{G}r(n)$ is the set of Eil graphs on n vertices whose vertices are labeled by generators, which we indicate by using label i in place of x_i.

We now develop the pairing between $\mathcal{G}r(n)$ and $\mathcal{T}r(n)$, as first developed in [Sin, Sin13], arising in the study of configuration spaces.

Definition 4.13. Let the height of a vertex in a tree be the number of edges between that vertex and the root, and let $gcv(i,j)$ be the vertex of greatest height which lies beneath leaves labelled i and j

Given $G \in \mathcal{G}r(n)$ and $T \in \mathcal{T}r(n)$, define the map

$$\beta_{G,T} : \big\{\text{edges of } G\big\} \longrightarrow \big\{\text{internal vertices of } T\big\}$$

by sending the edge $\overset{j}{\underset{i}{\nearrow}}$ in G to the vertex $gcv(i,j)$ in T. The configuration pairing of G and T is

$$\big\langle G,\, T\big\rangle = \begin{cases} \displaystyle\prod_{\substack{e \text{ an edge} \\ \text{of } G}} \operatorname{sgn}\big(\beta_{G,T}(e)\big) & \text{if } \beta \text{ is surjective,} \\ 0 & \text{otherwise,} \end{cases}$$

where $\operatorname{sgn}\left(\beta\Big(\overset{j}{\underset{i}{\nearrow}}\Big)\right) = 1$ if leaf i is to the left of leaf j in the embedding of T; otherwise it is -1.

The first main result of this section is that this pairing also governs our letter-linking invariants in the case where the symbols and commutators have each generator occurring only once.

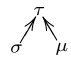
Theorem 4.14. *Let $G \in \mathcal{G}r(n)$ and $w \in UC(n)$. Then $\Phi_G(w) = \langle G, T_w\rangle$.*

Proof. If $\langle G, T_w\rangle = \pm 1$ we argue inductively, at first not tracking signs. The theorem is immediate when $n = 1$. If $n > 1$, set $w = [w_1, w_2]$. As $\langle G, T_w\rangle = \pm 1$ then there is a unique edge e of G such that $\beta_{G,T_w}(e) = v$. Removing e from G yields two acyclic graphs G_1 and G_2, whose vertex labels must coincide with those of T_1 and T_2 respectively, since any other edge connecting a vertex with label among those in T_1 to a vertex with label from T_2 would also have its image under β_{G,T_w} equal to v. Moreover, we must have $\langle G_i, T_i\rangle = \pm 1$ for $i = 1, 2$, which inductively implies $\Phi_{G_i}(w_i) = \pm 1$.

Now choose to reduce of G so that the vertices of e are the last two and then, say, choose the vertex in G_1 for the last reduction. In this case, the symbol for $\rho_V G$ will be $(\sigma_1)\sigma_2$, where σ_i is the symbol reduction of G_i. Because the generators which occur in w_1 and w_2 are distinct, $\Lambda_{\sigma_1}(w_1 w_2 {w_1}^{-1} {w_2}^{-1})$ occurs only in the w_1 and ${w_1}^{-1}$ sub-words. The occurrences of the free letter of σ_1 will occur in pairs across w_1 and ${w_1}^{-1}$ mapped to each other by the canonical involution, with a total multiplicity of such pairs of ± 1. We cobound according to this choice of pairs. Similarly $\Lambda_{\sigma_2}(w)$ will only occur in w_2 and ${w_2}^{-1}$ sub-words. Only the occurrences in w_2 will intersect the cobounding, and by the inductive assumption that $\Phi_{G_i}(w_i) = \pm 1$ we have $\Phi_{\pm(\sigma_1)\sigma_2} = \pm 1$.

We obtain the signed result by noticing that $\langle G, T_w\rangle = \varepsilon \langle G_1, T_{w_1}\rangle\langle G_2, T_{w_2}\rangle$, where $\varepsilon = 1$ if the initial vertex of e is in G_1 or -1 if its initial vertex is in G_2. Choose the last vertex for reduction to be the initial vertex of e, so that the result will be $(\sigma_1)\sigma_2$ if e points from G_1 to G_2 or $\sigma_1(\sigma_2)$ otherwise. The first case was chosen above, and we now have a signed equality $\Phi_G(w) = \Phi_{G_1}(w_1)\Phi_{G_2}(w_2)$. In the second case, it is the elements of $\Lambda_{\sigma_1}({w_1}^{-1})$ which occur between pairs of $\Lambda_{\sigma_2}(w_2^{\pm 1})$ so we have $\Phi_G(w) = -\Phi_{G_1}(w_1)\Phi_{G_2}(w_2)$. Both cases agree with the inductive formula for $\langle G, T_w\rangle$.

If $\langle G, T_w\rangle = 0$, there is a vertex v with $\beta_{G,T_w}(e) = \beta_{G,T_w}(f) = v$ for at least two edges e and f. Reduce G so that the images in the quotient of these two edges are the last two edges, with remaining symbols

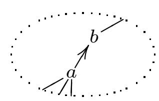
σ, τ, μ. That is, reduce, up to edge orientation, to 

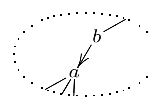
. The final reduction can thus be chosen as $\pm(\sigma)\tau(\mu)$. Because $\beta_{G,T_w}(e) = \beta_{G,T_w}(f) = v$, the free letters of σ and μ occur in w within a commutator which is later commuted with the free letter of τ. The coboundings of free letters of σ and μ can be chosen within this first commutator, and thus disjoint from the free letter of τ, implying $\Phi_G(w) = 0$. □

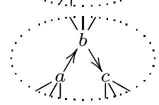
4.3. **Reduction of Graphs.** The results of the previous section are "at the level of operads." We expand our consideration to the (co)free (co)algebras built from them. As needed for our work in group theory, we continue to build from a generating set rather than from a vector space.

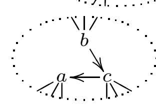
Definition 4.15. Let $\mathcal{E}il(i)$ be the quotient of the span of $\mathcal{G}r(i)$ by antisymmetry and Arnold relations which are shown below.

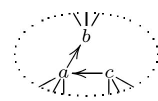
Let $\mathbb{E}_n$ be the quotient of the span of Eil graphs of any size with vertices are labelled by the generating set $x_1, \cdots, x_n$, by antisymmetry and Arnold relations

(arrow-reversing) 

$= -$ 

(Arnold) 

$+$ 

$+$ 

$= 0,$

along with setting graphs with cycles to zero.

If we let W be the span of $x_1, \cdots, x_n$, then $\mathbb{E}_n \cong \bigoplus_i \mathcal{E}il(i) \otimes_{\mathcal{S}_i} W^{\otimes i}$, where the symmetric group $\mathcal{S}_i$ acts on $\mathcal{E}il(i)$ by permuting vertex labels and on $W^{\otimes i}$ as usual by permuting factors. Recall that the free Lie algebra $\mathbb{L}_n$ is isomorphic to $\bigoplus_i \mathcal{L}ie(i) \otimes_{\mathcal{S}_i} W^{\otimes i}$. A key result of [SW11], namely its Corollary 3.11, is the following.

Theorem 4.16. *The cofree Lie coalgebra $\mathbb{E}_n$ pairs perfectly with $\mathbb{L}_n$ through the extension of the configuration pairing between all $\mathcal{E}il(i)$ and $\mathcal{L}ie(i)$ and the Kronecker pairing of W with itself extended to $W^{\otimes i}$.*

The graphs which define $\mathbb{E}_n$ include our symbol graphs $\mathcal{S}G_{0,n}$, but in general are not required to have distinct generators labeling the endpoints of any edge.

Definition 4.17. An edge in a labeled graph is called homogeneous if it connects vertices with the same label and heterogeneous otherwise.

Using this terminology, our previously defined distinct-vertex Eil graphs are those for whom which all edges are heterogeneous. In [SW11] it is noted that linear graphs span $\mathbb{E}_n$, and in [WS16] a new basis is constructed using "star graphs". A different spanning set is crucial for our work.

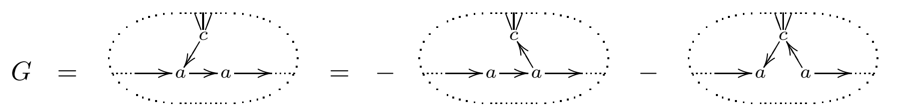
Theorem 4.18. *The distinct vertex Eil graphs span $\mathbb{E}_n$ over the rational numbers.*

Proof. We use the defining antisymmetry and Arnold identities to express any graph in $\mathbb{E}_n$ as a linear combination of graphs with fewer homogeneous edges, yielding the result by induction The reduction process is delicate since we will have recurrent appearances of terms, so care with signs is essential.

Consider any representative graph in $\mathbb{E}_n$. Pick any (co)generator a of the cofree Lie coalgebra which appears in the graph and consider a maximal connected subgraph consisting entirely of edges connecting that generator to itself. As is standard, see for example [Sin13, Sin], the Arnold identity applied to that subgraph can be used to reduce valence and ultimately yield a linear combination of linear graphs. Because we apply identities exclusively to edges in the subgraph, these will have the same number of homogeneous edges.

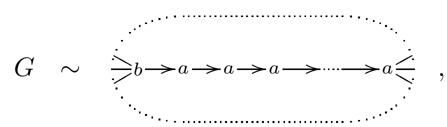
The resulting graphs will each have a linear homogeneous subgraph all of whose vertices except those at the ends have two homogeneous edges. All other edges incident to these vertices, which we call normal edges since they connect outside the subgraph, are heterogeneous.

We show such a graph G with a linear homogeneous subgraphcan be reduced. We first "move" the normal edges to one end of the linear graph (in our pictures, to the right) using the Arnold identity as follows:



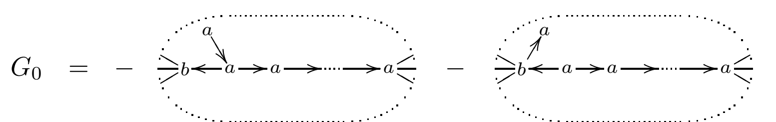
Note that there may be other edges connected to these vertices that are not drawn in the picture. No homogenous edges have been added in any graph. In the middle graph, c is connected to an a which is one further to the right, as desired. In the third graph we have decreased the number of homogeneous edges by one.

Reserving $=$ for equality in $\mathbb{E}_n$ (that is, modulo Arnold and anti-symmetry relations), we let $\sim$ denote equivalence in $\mathbb{E}_n$ modulo graphs with fewer homogeneous edges, which are reducible by induction hypothesis. Applying the above identity repeatedly, we have

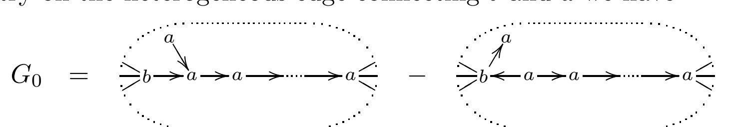


where there could be single normal edge on the left-most a, multiple possible on far right a, and no normal edges on the "middle" a's.

We call the graph on the right-hand side G_0 and now reduce it. Apply the Arnold identity to the $b \to a \to a$ subgraph to get

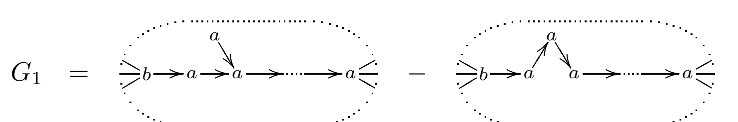


Using antisymmetry on the heterogeneous edge connecting b and a we have



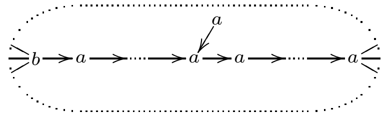
Let G_1 be the first graph on the right hand side. The second term on the right hand side has fewer homogeneous edges, so $G_0 \sim G_1$.

We now "transfer the normal $a \to$ in G_1 down the graph to the end." We first apply the Arnold identity to this normal edge and the one following it in the linear chain, and then anti-symmetry to the resulting terms to obtain



Call the first graph on the right hand side G_2, and notice the last graph is G_0. So $G_1 = G_2 - G_0$, and thus $2G_0 \sim G_2$.

In general, let

$G_i \quad = $ 

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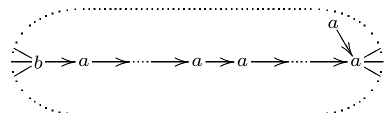
where the "normal" $a \to$ is connected to the ith a in the linear chain. We argue as above, applying the Arnold identity to the normal edge and edge which follows it to deduce $G_i = G_{i+1} - G_0$. Hence for each $i < n$, where n is the length of the linear chain of a's,

$$G_1 = G_2 - G_0 = G_3 - 2G_0 = \cdots = G_i - (i-1)G_0.$$

We thus focus on

$G_n \quad = $ 

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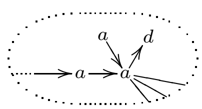
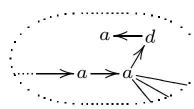
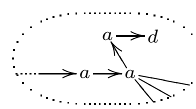
working at the right end of the graph to finish our argument. Apply the Arnold identity to each of the heterogeneous edges originating at the final a in the chain along with the homogeneous edge at the end, followed by anti-symmetry to obtain graphs with arrows "moving right." The first step is

$G_n \quad = $ 

$= \; -$ 

$-$ 

.

The first graph on the right hand side has fewer homogeneous edges. We then similarly apply the Arnold identity to the second graph on the right hand side, using another normal heterogeneous edge which emanates from final a in the (original) linear chain, followed again by anti-symmetry. Doing so for all of these normal edges, we deduce that $G_n \sim -G_0$. Thus, we have $G_1 = G_n - (n-1)G_0 \sim -nG_0$. Since $G_0 \sim G_1$ and $G \sim G_0$ we deduce that $(n+1)G$ is equivalent to a linear combination of graphs with fewer homogeneous edges, completing the reduction argument. The base case of no homogeneous edges is a tautology. □

See Section A.3 for an example of the reduction given by the proof of this theorem.

5. The main theorem, and connections

5.1. The main theorem. Recall Corollary 3.13 that our letter linking invariants are well-defined on lower central series subquotients. Our main result in this paper is the following.

Theorem 5.1. *The homomorphisms* $\Phi : \mathbb{Q}\mathcal{S}ymb_i \to \mathrm{Hom}(\gamma_i F_n / \gamma_{i+1} F_n, \mathbb{Q})$ *are surjective.*

Before proving this, we need a last calculational tool, parallel to the operadic approach to free Lie algebras. We relate values of our letter linking invariants under homomorphisms induced by set maps of generators.

Definition 5.2. Let F_S denote the free group on the generating set S. Let $f : S \to T$ be a map of generating sets, and let f_* denote the induced homomorphism on free groups as well as the induced map on the set of symbols. Let Σ_f denote the automorphisms of S which commute with f.

The automorphisms Σ_f are isomorphic to a product of symmetric groups.

Theorem 5.3. *Let* $f : S \to T$ *be a map of (generating) sets,* $w \in UC(n)$ *and* σ *a symbol on* S *of depth* n. *Then*

$$\Phi_{f_*\sigma}(f_* w) = \sum_{p \in \Sigma_f} \Phi_{p \cdot \sigma}(w).$$

We use this to understand letter linking invariants with repeated letters, relating them to those with unique letters, which are understood through Theorem 4.14.

Proof of Theorem 5.3. We first show that

$$\Phi_{f_*\sigma}(f_*w) = \sum_{\widetilde{\sigma} | f_*(\widetilde{\sigma}) = f(\sigma)} \Phi_{\widetilde{\sigma}}(w).$$

The sum contains the sum named in the theorem, along with additional terms which we will show vanish.

We prove this equality through analysis of lists, showing inductively that $\Lambda_{f_*\mu}(f_*w) \cong \bigcup \Lambda_{\widetilde{\mu}}(w)$ – that is, that these are in bijective correspondence respecting f – where the union is over $\widetilde{\mu}$ such that $f_*(\widetilde{\mu}) = f_*(\mu)$. For μ of depth zero, that is lists of occurrences of some generator, this is immediate. Suppose this equality of lists holds for μ_1 and μ_2 of depth less than n. By Theorem 3.12 all of the $\Phi_{\widetilde{\mu_1}}(w)$ vanish, so we may choose all $d^{-1}\Lambda_{\widetilde{\mu_1}}(w)$. Through our inductive bijection, the images of these under f_* gives a choice of $d^{-1}\Lambda_{f_*\mu_1}(f_*w)$. Moreover, each intersection of $\Lambda_{f_*\mu_2}(f_*w)$ with this cobounding corresponds to the intersection of some $\Lambda_{\widetilde{\mu_2}}(w)$ with a $d^{-1}\Lambda_{\widetilde{\mu_1}}(w)$. Through the bijective correspondence of product of the set of $\widetilde{\mu_1}$ over μ_1 with the $\widetilde{\mu_2}$ over μ_2 with the set of $\widetilde{(\mu_1)\mu_2}$ over $(\mu_1)\mu_2$, we establish our inductive step that $\Lambda_{f_*(\mu_1)\mu_2}(f_*w) \cong \bigcup \Lambda_{\widetilde{(\mu_1)\mu_2}}(w)$ and thus our first equality.

To deduce the equality of the theorem we see that $\sum_{\widetilde{\sigma}|f_*(\widetilde{\sigma})=f(\sigma)}$ which are not of the form $p \cdot \sigma$ for $p \in \Sigma_f$ must have some repeated letter. But $w \in UC(n)$, so there will be at least one letter which occurs in w but not $\widetilde{\sigma}$. That letter can then be removed from w without changing $\Lambda_{\widetilde{\sigma}}$. But removing the letter from w is equivalent to replacing the letter by the identity element. Since w is a commutator the resulting word would represent the identity element. □

We now extend Theorem 4.14 from graphs with unique vertex labels to all distinct-vertex graphs.

Definition 5.4. Let $w \in \gamma_i F_n$, and let W be the span of generators of F_n. We set set $\lambda(w) \in \mathbb{L}_n \cong \mathcal{L}ie(i) \otimes W$ to be the image of w in $\gamma_i F_n / \gamma_{i+1} F_n$, composed with its isomorphism with the ith graded component of the free Lie algebra.

Explicitly, $\lambda(w)$ converts a commutator to the corresponding Lie bracket. Theorem 5.3 leads to the following.

Corollary 5.5. *$\Phi_G(w) = \langle G, \lambda(w) \rangle$, where $\langle -, - \rangle$ denotes the configuration pairing.*

Proof. Let $\tilde{w}$ be a word so that $f_*(\tilde{w}) = w$ for some map of generating sets S. For $\Phi_G(w)$ to be non-zero there must be a $\widetilde{G}$ with $f_*(\widetilde{G}) = G$. Theorem 5.3 then gives a formula for $\Phi_G(w)$. But we can apply Theorem 4.14 to every term in the right-hand side. Doing so we obtain terms in the definition of $\langle G, \lambda(w) \rangle$, which is he extension of the pairing between $\mathcal{E}il(i)$ and $\mathcal{L}ie(i)$ and Kronecker pairing on W to $\mathcal{E}il(i) \otimes_{\mathcal{S}_n} W^{\otimes i}$ and $\mathcal{L}ie(i) \otimes_{\mathcal{S}_n} W^{\otimes i}$. The terms in this extension which do not occur in the application of Theorem 5.3 will not contribute to this sum, as the Kronecker pairing will be zero. □

The proof of our main result is now a matter of assembly.

Proof of Theorem 5.1. By Proposition 4.9, any symbol $\sigma \in \mathcal{S}ymb_n = \mathcal{S}G_{n,0}$ is the reduction of some graph G in $\mathcal{S}G_{0,n}$. By Corollary 5.5, the values of Φ_σ on $\gamma_i F_n$ coincide with the configuration pairing of G on the i-graded summand of $\mathbb{L}_n$. By Theorem 4.18, configuration pairings with distinct-vertex graphs span the functionals given by all graphs. By Corollary 3.3 of [SW11] pairing with all such graphs modulo Arnold and anti-symmetry is perfect on this i-graded summand, which is isomorphic to $\gamma_i F_n / \gamma_{i+1} F_n$. □

5.2. **Comparison with Fox derivatives.** There is already a well-known collection of homomorphisms which span the linear dual of the lower-central Lie algebra, namely those given by Fox's free differential calculus [Fox53, Lyn58], whose definition we recall below. These differ from the functionals we provide in substantial ways.

- Fox derivatives span homomorphisms to the integers, while letter-linking homomorphisms only span over the rationals. See Section A.2.
- Fox derivatives are defined on the entire free group, while letter-linking homomorphisms are only defined on subgroups.
- As shown below, Fox derivatives correspond to evaluation of the linear graph spanning set for the cofree Lie coalgebra $\mathbb{E}_n$, while letter-linking homomorphisms correspond to evaluation of the distinct-vertex spanning set.
- Fox derivatives, as developed in part by Chen, Fox and Lyndon [Lyn58], are more immediately compatible with the Chen model of rational homotopy theory while letter-linking homomorphisms are drawn from the Quillen model.
- For hand calculations, letter-linking numbers involve fewer calculations. See Section A.1.
- For fundamental groups of punctured surfaces, letter linking invariants immediately give rise to lower bounds on the complexity of curves which represent elements of $\gamma_i F_n$.

With our eyes towards applications to mapping class groups (first author) and non-simply connected rational homotopy theory (second author) we believe the first two properties in which letter linking homomorphisms are inferior with respect to Fox derivatives are worth trading for the properties in which they are superior.

We now make the connection between Fox derivatives and our model for cofree Lie coalgebras, starting with the definition of the former.

Definition 5.6. Let F_n be the free group on n generators and let $\alpha : \mathbb{Z}[F_n] \to \mathbb{Z}$ be the augmentation. A derivation D is a map $D : \mathbb{Z}[F_n] \to \mathbb{Z}[F_n]$ such that

(1) $D(u + v) = Du + Dv$
(2) $D(uv) = Du \cdot \alpha(v) + u \cdot Dv$

Theorem 5.7. [Fox53] *Let $x_1, \ldots, x_n$ denote the generators of the free group F_n. There is a unique derivation*

$$\frac{\partial}{\partial x_i} : \mathbb{Z}[F_n] \to \mathbb{Z}[F_n]$$

such that $\frac{\partial}{\partial x_i}(x_j) = \delta_{i,j}$, the kronecker delta. We call this the derivative, or Fox derivative, with respect to x_i and denote it by ∂_{x_i}.

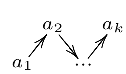
This derivation is then iterated.

Definition 5.8. Let $c = a_1, \ldots, a_k$ be some collection of the generators $x_1, \ldots, x_n$ where repeats are allowed. Then for $v \in \mathbb{Z}[F_n]$ inductively define

$$\partial_{a_1 \cdots a_k}(v) = \partial_{a_1}(\partial_{a_2 \cdots a_k}(v)).$$

Define $\partial_c^\circ(v)$ to be $\alpha(\partial_c(v))$.

In [Lyn58] the others produce a collection of $c = a_1 \cdots a_k$ so that ∂_c° form a basis for the dual space of each $\gamma_i F_n / \gamma_{i+1} F_n$. We produce a new proof of this fact in order to compare it with the present work, starting with the analogue of Corollary 5.5.

Theorem 5.9. *Let $c = a_1 \cdots a_k$ and let G_c be the graph* 

, and let $w \in \gamma_k F_n$. Then $\partial_c^\circ w = \langle G_c, \lambda(w) \rangle$.

Proof. Equation (3.3) of [Lyn58] states that for $u \in \gamma_i F_n$, $v \in \gamma_j F_n$ with $i + j = k$,

$$\partial^\circ_c([u, v]) = \partial^\circ_{c_f}(u)\partial^\circ_{c_l}(v) - \partial^\circ_{c'_l}(u)\partial^\circ_{c'_f}(v),$$

where $c_f = a_1 \cdots a_i$, $c'_f = a_1 \cdots a_j$, and c_l and c'_l are their complements in c. We compare this equality with the bracket-cobracket formula $\langle G_c, \lambda([u, v])\rangle =]G_c[u \otimes v$, established in Corollary 3.14 of [SW11]. These formulae determine the values of the Fox derivatives and graph coalgebra pairings.

Since $]G_c[= G_{c_f} \otimes G_{c_l} - G_{c'_l} \otimes G_{c'_f}$, we may use these formulae to establish the theorem inductively, starting with the weight zero case which is immediate. □

As developed in [Lyn58], the bracket-cobracket reduction formula for Fox derivatives then shows that they yield the coefficients of the map from the free Lie algebra to its universal enveloping algebra, also known as the tensor algebra. As noted in Remark 1.5 of [Wal], the resulting functionals are represented by the linear graph spanning set, or basis if one choses a subset of linear graphs such as Lyndon-Shirsov words. We show in Section A.2 of the Appendix that the spanning set for linear functions we develop, represented by distinct vertex graphs, is distinct from this classical spanning set.

5.3. **Further directions.** We already have planned applications for this work, namely to the study of the Johnson homomorphisms for mapping class groups by the second author. But we hope our new insight into the lower central series Lie algebra of free groups, first studied by Magnus eighty years ago [Mag40], will have impact in a few directions.

In algebra, the immediate question is whether and how letter linking invariants could be generalized to arbitrary finitely presented groups. For example, the fundamental group of the genus-two surface has four generators $a, \ldots d$ and the relation $[a, b] = [c, d]$. We conjecture there is a complete collection of letter linking invariants which now include the linear combination $\Phi_{(a)b} + \Phi_{(c)d}$, but neither count on its own. We can see this invariant in our formalism as follows. The Lie coalgebrac bar complex on the cochains of a space provides the setting for Hopf invariants in higher dimensions [SW13]. In this paper the space in question has been a wedge of circles, who cochains are equivalent to the first cohomology, resulting in the bar complex being equivalent to the cofree Lie coalgebra $\mathbb{E}_n$. In this setting of a surface, we still have formality, with the cohomology generated by classes A, B, C, D, say Kroncker dual to the homology classes of $a, \ldots, d$, with the relation $AB = -CD$. In the bar complex, $\overset{B}{\underset{A}{\nearrow}}$ will not be a cycle, having coboundary AB, but $\overset{B}{\underset{A}{\nearrow}} + \overset{D}{\underset{C}{\nearrow}}$ will be a cycle and will reduce to $\Phi_{(a)b} + \Phi_{(c)d}$. We expect the Lie coalgebraic bar complex to control these letter linking invariants in general.

If we can use such bar complexes to produce the dual to the lower central series Lie algebra of a group, one could try to merge this with the Lie coalgebraic models used in the simply connected setting. A primary issue to resolve is that the notion of distinct vertices which is essential to defining letter-linking invariants does not have a natural counterpart in higher dimensions where the constructions naturally start with vector spaces or chain complexes rather than spanning sets. If such models can be developed, they should be compared with new Lie models of Buijs-Félix-Murillo-Tanré [BFMT18] which are based on the Lawrence-Sullivan cylinder object [LS14].

There are plenty of elementary questions as well. While we know that distinct-vertex graphs span cofree Lie coalgebras on a set of (co)generators, we have yet to find a basis. It would be interesting to connect such a basis, as well as closer analysis of relations, to the literature on (distinctly) colored trees. Looking at the examples in Section A.2, it seems likely that understanding the values of that basis on free Lie algebras could lead to new bases for the latter. These examples also point to the question of computing the indices of the functionals arising from letter-linking invariants within all integer-valued functionals. One should decompose the free Lie algebra on a generating set into summands by the number of times each generator occur and compute on those summands, in which case so far we only see factorials arise.

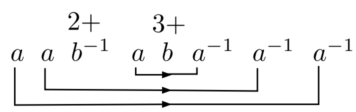
Appendix A. Examples

A.1. **Letter linking invariants and Fox derivatives.** The geometric inspiration for or our linking invariants gives rise to visual algorithm to computing these numbers by hand.

Let w be a word with lists $L_a$and L_b with $\Phi_a(L_a) = 0$. We calculate $d^{-1}L_a \wedge L_b$ through a diagram, in the following steps:

(1) For each (a, ϵ) in L_a, place a + sign above the corresponding a in w if $\epsilon = 1$ and a minus sign if $\epsilon = -1$, or just list total multiplicities over the letters.
(2) Each interval chosen for $d^{-1}L_a$ corresponds to a choice of elements with opposite total sign. For each, draw an arrow originating under the letter with positive total sign and ending under the element with negative total sign.
(3) As in the first step, decorate each b in w with its multiplicity in L_b, say m which is the positive occurrences minus the negative occurrences of this b in the list. The for each b consider the arrows which "pass through" it. If there are p arrows heading left to right and if there are q arrows heading from right to left which cross that occurrence of b, then replace its multiplicity by $m(p - q)$. If there are no arrows are underneath the letter replace its multiplicity by 0. These multiplicities define $d^{-1}L_a \wedge L_b$.

Consider for example $w = [aab^{-1}, a]$. As the theory dictates, we see that the linking numbers $\Phi_a(w)$ and $\Phi_b(w)$ vanish, as do $\partial_a^\circ(w)$ and $\partial_b^\circ(w)$. To calculate $\Phi_{(a)b}(w)$, our diagrammatic approach yields



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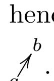
which means $\Phi_{(a)b}(w) = 2 \cdot \operatorname{sign}(b^{-1}) + 3 \cdot \operatorname{sign}(b) = 1$.

To compute ∂_{ab}, we first apply ∂_b:

$$\partial_b(w) = -aab^{-1} + aab^{-1}a.$$

Applying ∂_a gives

$$\partial_a(aab^{-1}) = 1 + a$$
$$\partial_a(aab^{-1}a) = 1 + a + aab^{-1}.$$

Thus $\partial_{ab}(w) = aab^{-1}$ and hence $\partial_{ab}^\circ(w) = 1$, so $\Phi_{(a)b}(w) = \partial_{ab}(w)$ as follows from their values both corresponding to the graph 

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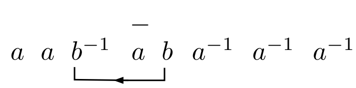
The same invariant through different symbols or derivatives leads to different calculations. The diagrammatic computation of $\Phi_{(b)a}(w)$ is



,

which gives $\Phi_{(b)a}(w) = -1 \cdot \operatorname{sign}(a) = -1$. Computing $\partial_{ba}(w)$ we first have

$$\partial_a(w) = 1 + a + aab^{-1} - aab^{-1}aba^{-1} - aab^{-1}aba^{-1}a^{-1} - aab^{-1}aba^{-1}a^{-1}a^{-1}.$$

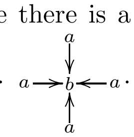
Then apply ∂_b to get

$$\begin{aligned}\partial_b(1) = \partial_b(a) &= 0;\\ \partial_b(aab^{-1}) &= -aab^{-1};\\ \partial_b(aab^{-1}aba^{-1}) &= -aab^{-1} + aab^{-1}a;\\ \partial_b(aab^{-1}aba^{-1}a^{-1}) &= -aab^{-1} + aab^{-1}a;\\ \partial_b(aab^{-1}aba^{-1}a^{-1}a^{-1}) &= -aab^{-1} + aab^{-1}a.\end{aligned}$$

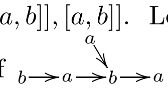
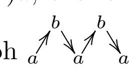
After cancellation we arrive at $\partial_{ba}(w) = 2aab^{-1} - 3aab^{-1}a$ and hence $\partial^{\circ}_{ba}(w) = -1$.

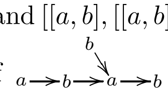
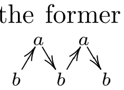
We see that the diagrammatic method can lead to substantially easier work by hand, moreso as the depth increases. We conjecture letter linking invariants are more efficient than Fox derivatives computationally.

A.2. **Values on a Free Lie algebra basis.** We choose a basis for our letter-linking invariants and share its values on a choice of Hall basis, which is also the Lyndon basis, for $\gamma_5 F_2/\gamma_6 F_2$. This pairing decomposes into blocks, according to number of times each generator, which we call a and b, occur.

At the extremes, we have the $[a,[a,[a,[a,b]]]]$, the only basis element with four a's. Here there is a unique linking invariant symbol, $(a)(a)(a)(a)b$, which is reduction of the distinct-vertex graph. .

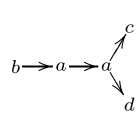
The value of the invariant is 24. The case of only one a and four b's is similar.

With three a's and two $b's$ there are Hall basis elements $[a,[a,[[a,b],b]]]$ and $[[a,[a,b]],[a,b]]$. Letter linking symbols are $(((b)a)(a)b)a$ and $((((a)b)a)b)a$, the former being the reduction of  and the latter being the reduction of the "linear" graph . The pairing here is not a Kronecker pairing, being represented by the matrix $\begin{bmatrix}4 & -2\\ 4 & 4\end{bmatrix}$.

Next, we have Hall basis elements $[a,[[[a,b],b,],b]]$ and $[[a,b],[[a,b],b]]$ and letter linking symbols $(((a)b)(b)a)b$ and $((((b)a)b)a)b$, the former being the reduction of  and the latter being the reduction of the "linear" graph . Here the pairing represented by the matrix $\begin{bmatrix}6 & -2\\ 0 & 4\end{bmatrix}$, yielding the same determinant (index) as in the previous case.

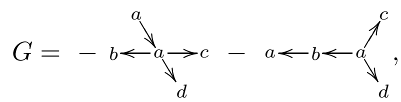
Other choices for representative letter-linking invariants give the same results, up to sign. Thus the Hall basis, which in this case is also the Lyndon basis, is not Kronecker in pairing with letter-linking invariants. It would be interesting to see such a dual basis in general, since it seems like it would have symmetry properties which bases which use orderings on the generating set, both classical bases as well as new one such as those in [WS16], do not have.

A.3. **Reduction to distinct vertex Eil graphs.** We reduce the graph

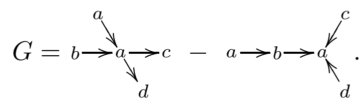
$G =$ 

to a rational linear combination of distinct vertex graphs, following the procedure outlined in the proof of Theorem 4.18.

There is only one maximal subgraph of a's and it is already linear. Thus, the first step is applying Arnold to the first two edges to get

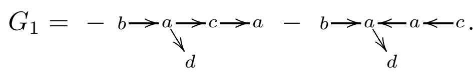


which applying antisymmetry implies

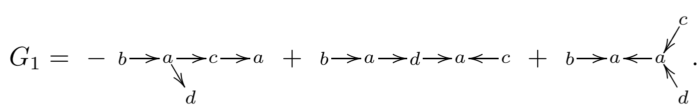


In the notation of the proof of the Theorem 4.18, the second graph above is the $H$and the first graph is G_1, which in this case is also our G_n. So we have $G = G_1 - H$.

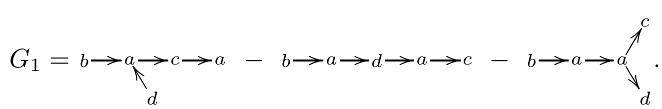
In the first graph we just need to apply the Arnold relation twice at the right end of the graph. The first Arnold relation, and redrawing the graph, gives



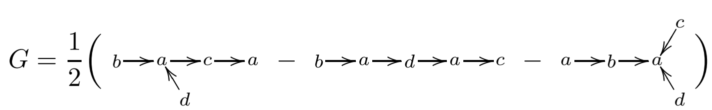
The first graph is a distinct vertex graph, which would have been called Γ_1 in the proof. We apply the Arnold identity to the second graph to get



Rewriting using the antisymmetry relation,



We then plug this into $G = G_1 - H$ and solve for G to get



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