

MOD-TWO COHOMOLOGY RINGS OF ALTERNATING GROUPS

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ABSTRACT. We calculate the mod-two cohomology of all alternating groups together, with both cup and transfer product structures, which in particular determines the additive structure and ring structure of the cohomology of individual groups. We show that there are no nilpotent elements in the cohomology rings of individual alternating groups. We calculate the action of the Steenrod algebra and discuss individual component rings. A range of techniques is needed: an almost Hopf ring structure associated to the embeddings of products of alternating groups, the Gysin sequence relating the cohomology of alternating groups to that of symmetric groups, Fox-Neuwirth resolutions, and restriction to elementary abelian subgroups.

1. INTRODUCTION

Alternating groups are a fundamental series of simple groups whose cohomology has remained mysterious, even additively, for over fifty years after the mod- p cohomology of symmetric groups was determined additively by Nakaoka [14]. We present their mod-two cohomology in Theorem 8.1, giving generators and relations using cup product as well as a restriction coproduct and transfer product associated to the standard embedding of a product of alternating groups in a larger one.

We use the Gysin sequence relating cohomology of alternating and symmetric groups to find additive bases. We use resolutions based on geometric ideas of Fox and Neuwirth [6, 9] to produce generating cohomology classes. We show that our product and coproduct structures comprise an “almost Hopf ring” structure, and use that structure along with a canonical involution to propagate cohomology and establish a basic framework of relations. Such a suite of product and coproduct structures forms a Hopf ring in the setting of symmetric groups, as first developed by Strickland and Turner [18]. Finally, we show that restriction to the cohomology elementary abelian subgroups is injective, and calculate these restriction maps to establish a full set of relations. Our detection result implies that there are no nilpotent elements in the cohomology of alternating groups, which resolves a long-standing open question. Throughout, our understanding of the cohomology of symmetric groups as a Hopf ring [8] provides essential input.

The presentations of cohomology for alternating and symmetric groups – Theorems 8.1 and 3.1 – are parallel. The generators predominantly map to each other in the Gysin sequence. There is a notion of level for generators (the ℓ in $\gamma_{\ell,m}$), and while the description of cohomology for symmetric groups is uniform, that of alternating groups is irregular for small levels. At level three or greater, there are two Hopf ring generators for alternating groups for each generator for symmetric groups. These two generators map to one another under conjugation and annihilate each other under cup product. This structure is unstable, in that only the sum of such pairs lift to larger alternating groups. At level two, there are still two sets of generators, but instead of annihilating each other under cup product there are exceptional relations which start at \mathcal{A}_4 and then propagate, as determined by coproduct structure. Finally, at level one only one set of generators occurs, but we need a separate set of generators for each component. In comparison with symmetric groups that set lacks the degree-one generator, which is the Euler class in the Gysin sequence.

Transfer products are relatively simple, with relations mostly governed by a notion of charge which implies that the transfer product of neutral classes vanishes. Cup products are complicated, with complexity driven by both basic relations and Hopf ring structure, in contrast to the setting of symmetric groups where the latter alone accounts for all of the multiplicative complexity.

Much as ring structure determines additive structure, our description using two products determines cup product ring structure alone, for example yielding an elementary algorithm for finding ring generators. We carry out a calculation for \mathcal{A}_8 , and then explain why techniques which yield more global results for symmetric groups do not apply in this setting. We also calculate Steenrod operations on our generators, which determines the global structure.

We develop our four distinct techniques for understanding the cohomology of alternating groups in the next four sections, before applying them for calculations in the last five sections. In comparison with symmetric groups, alternating groups present substantial technical challenges at each step of calculation. We thank Alejandro Adem and Paolo Salvatore for helpful comments throughout our time working on this project.

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2. PRODUCT AND COPRODUCT STRUCTURES ON SERIES OF GROUPS

Definition 2.1. A product series of finite groups is a collection $\{G_i\}_{i \geq 0}$ with embeddings $e_{n,m} : G_n \times G_m \hookrightarrow G_{n+m}$ which are associative and commutative up to conjugation.

Examples include symmetric groups, general linear groups over finite fields, and series of Coxeter groups. We are concerned with mod-two cohomology of alternating groups, so we can set $G_i = \mathcal{A}_{2i}$, yielding a product series. For the rest of this paper, when we refer to \mathcal{A}_n we assume n is even, generally making this explicit. For work on odd alternating groups commutativity would fail up to up to conjugation, so the lack

of commutativity on cohomology would be controlled by a standard involution, which plays a substantial part in the mod-two setting as well.

Definition 2.2. An almost-Hopf ring is a vector space V with two associative, commutative products \odot and \cdot , and a coproduct Δ so that (\cdot, Δ) defines a bialgebra, and \cdot distributes over \odot with respect to the coproduct. Explicitly, distributivity means the following diagram commutes, where μ_\odot and μ_\cdot are the bilinear maps which correspond to the multiplications and τ is the twist map that exchanges the middle two factors.

$$\begin{array}{ccccc} V^{\otimes 3} & \xrightarrow{\tau \circ (\Delta \otimes id)} & V^{\otimes 4} & \xrightarrow{\mu_\cdot \otimes \mu_\cdot} & V \otimes V \\ \downarrow id \otimes \mu_\odot & & & & \downarrow \mu_\odot \\ V \otimes V & \xrightarrow{\mu_\cdot} & & & V \end{array}$$

In formulas, distributivity means

$$a \cdot (b \odot c) = \sum_{\Delta a = \sum a'_i \otimes a''_i} (a'_i \cdot b) \odot (a''_i \cdot c).$$

Distributivity facilitates inductive calculations, especially in the graded setting, and implies that there are bases of the form $p_1 \odot p_2 \odot \cdots \odot p_i$ where each p_i is a product with respect to the \cdot multiplication. We call such a Hopf monomial basis, and we call the p_i the constituent \cdot -monomials.

A Hopf ring is a ring object in the category of coalgebras, which entails all of the above and a Hopf algebra structure for (\odot, Δ) , as well as an involution. (We do not know a categorical definition for almost Hopf rings.) Strickland and Turner [18] show that the generalized cohomology of symmetric groups forms a Hopf ring, which inspires the following. Recall that if H is a finite index subgroup of G the the induced map of the inclusion is a covering map, as seen clearly through the model $BH = EG/H \rightarrow BG = EG/G$.

Theorem 2.3. *The direct sum of cohomology with field coefficients of a product embeddings of finite groups, $\bigoplus_i H^*(BG_i, k)$, forms an almost Hopf ring where*

- \odot is the transfer map in cohomology associated to the cover $Be_{i,j} : BG_i \times BG_j \rightarrow BG_{i+j}$. We call this the transfer or induction product.
- \cdot is the cup product, defined to be zero between different summands.
- $\Delta_{i,j}$ is the natural map associated to the cover $Be_{i,j}$, and $\Delta = \bigoplus \Delta_{i,j}$.

Proof. Recall for example from Theorem II.1.9 of [2] that conjugation by G , on say the standard simplicial model of BG , induces the trivial map on group cohomology. Thus coassociativity and cocommutativity of the coproduct Δ are immediate from the assumption of associativity and commutativity of product maps up to conjugation.

As transfer maps commute with isomorphism of covering spaces, associativity and commutativity respectively of the transfer product follow from the fact that the conjugation isomorphisms between the two copies of $G_n \times G_m \times G_p$ (respectively $G_n \times G_m$) in G_{n+m+p} (respectively G_{n+m}) define isomorphisms of covering spaces of BG_{n+m+p} (respectively BG_{n+m}).

Because the coproduct is induced from a map of spaces, it forms a bialgebra with cup product.

For distributivity, consider the following diagram.

$$\begin{array}{ccccc} BG_i \times BG_j & \xrightarrow{D_{BG_i} \times D_{BG_j}} & BG_i \times BG_i \times BG_j \times BG_j & \xrightarrow{\tau \circ (e_{i,j} \times id)} & BG_{i+j} \times BG_i \times BG_j \\ \downarrow Be_{i,j} & & & & \downarrow id \times Be_{i,j} \\ BG_{i+j} & \xrightarrow{D_{BG_{i+j}}} & & & BG_{i+j} \times BG_{i+j}, \end{array}$$

where in general D_X denotes the diagonal map on X . The vertical maps are covering maps, and along with them the composite of the top horizontal maps and the bottom horizontal map define a pull-back of covering maps. Taking cohomology and applying natural maps horizontally and transfer maps vertically yields a commutative diagram because transfers commute with natural maps for pull-backs. This diagram coincides with the distributivity diagram of Definition 2.2 (reflected across a vertical line). \square

Because transfers exist in generalized cohomology theories, these structures translate to that setting, though as usual coproduct structures can be problematic when the ground ring is not a field. Strickland and Turner developed these structures to study Morava E -theory of symmetric groups [18]. In the case of K -theory of symmetric groups, which by the Atiyah-Segal theorem is the completion of their representation rings, the transfer product coincides with induction product, as first studied thoroughly by Zelevinsky [20].

We use extensively the fact that transfer maps for covering spaces commute with natural maps for pull-backs, especially for covering maps are between classifying spaces of finite groups. We record some standard facts about such pull-backs here.

Proposition 2.4. *For a finite group G and subgroups H, K a model for the pull-back*

$$\begin{array}{ccc} PB & \xrightarrow{g} & BK \\ f \downarrow & & \downarrow B\nu_K \\ BH & \xrightarrow{B\nu_H} & BG \end{array}$$

is given by $PB = (EG \times G)/H \times K$ where H acts by $h \cdot (e, g) = (he, hg)$ and K acts by $(e, g) \cdot k = (e, gk^{-1})$.

The maps from the pull-back are by defined by identifying for example $BK = EG \times G/G \times K$ where G acts diagonally.

The components of the pull-back are indexed by double-cosets $H \backslash G / K$, and the component indexed by HgK is $B(H \cap gKg^{-1})$.

For reference, we give a second proof of the result first established by Strickland and Turner [18].

Theorem 2.5. *The cohomology of symmetric groups with field coefficients is a Hopf ring, extending the almost Hopf ring structure of Theorem 2.3.*

Proof. After Theorem 2.3 we need only check that \odot and Δ form a bialgebra. For symmetric groups, the intersection of $\mathcal{S}_n \times \mathcal{S}_m$ with conjugates of $\mathcal{S}_i \times \mathcal{S}_j$ in \mathcal{S}_d (where $d = n + m = i + j$) are all possible $\mathcal{S}_p \times \mathcal{S}_q \times \mathcal{S}_r \times \mathcal{S}_s$ with $p + q = n$, $r + s = m$, $p + r = i$ and $q + s = j$. By Proposition 2.4, the following diagram is thus a pull-back square of covering spaces

$$\begin{array}{ccc} \sqcup BS_p \times BS_q \times BS_r \times BS_s & \longrightarrow & BS_n \times BS_m \\ \downarrow & & \downarrow \\ BS_i \times BS_j & \longrightarrow & BS_d. \end{array}$$

Starting at $H^*(BS_i \times BS_j)$ and mapping to $H^*(BS_n \times BS_m)$ by composing restriction and transfer in two ways, which agree because this is a pull-back, establishes the result. \square

For general product series of groups including alternating groups, the coproduct and transfer product do not define a bialgebra. Nonetheless, the two products bind the cohomology of the G_i , and distributivity provides control up to computability of the coproduct. We have such computability for alternating groups, as transfer product and coproduct are close to forming a bialgebra, as described in Theorem 3.21 whose proof is a modification of that of Theorem 2.5.

3. RELATIONSHIPS BETWEEN THE COHOMOLOGY OF ALTERNATING AND SYMMETRIC GROUPS

3.1. The cohomology of symmetric groups. Our foundation is a thorough understanding of cohomology of symmetric groups, the focus of [8] whose main result is the following.

Theorem 3.1. *As a Hopf ring, $\bigoplus_{n \geq 0} H^*(BS_{2n}; \mathbb{F}_2)$ is generated by classes $\gamma_{\ell, m} \in H^{m(2^\ell - 1)}(BS_{m2^\ell})$, with $\ell, m \geq 0$, where $\gamma_{0,0}$ is the unit for transfer product and $\gamma_{0,m}$ is the unit for cup product on component $2m$. The coproduct of $\gamma_{\ell, m}$ is given by*

$$\Delta \gamma_{\ell, m} = \sum_{i+j=m} \gamma_{\ell, i} \otimes \gamma_{\ell, j}.$$

Relations between transfer products of these generators are given by

$$\gamma_{\ell, n} \odot \gamma_{\ell, m} = \binom{n+m}{n} \gamma_{\ell, n+m},$$

which implies that the $\gamma_{\ell, 2^k}$ constitute a set of Hopf ring generators. Cup products of generators on different components are zero, and there are no other relations between cup products of Hopf ring generators.

We will regularly refer to this and other results from Sections 5 and 6 of [8]. In Section 6 of [8] we give a convenient graphical representation of the Hopf ring monomials basis which we call skyline diagrams. We do not use such here, but we suggest that readers translate to that language, and we indicate as we go along how our calculations would look using skyline diagrams. For example, the following notion describes the size of the “grounding blocks” of a diagram.

Definition 3.2. In the cohomology of symmetric groups, we say the scale of a cup product monomial is the maximum ℓ which occurs, with unit classes scale one by convention. The scale of a Hopf ring monomial is the minimum of the scales of its constituent cup monomials.

The scale is the smallest symmetric group which occurs in the image of a non-trivial iterated coproduct.

3.2. Restriction and transfer maps. As alternating groups are subgroups of symmetric groups, there are both restriction maps and transfers relating their cohomology.

Definition 3.3. If H is a subgroup of G with inclusion map ι understood we let res denote the natural restriction map $B\iota^*$ on cohomology and let tr denote the transfer map $B\iota^!$ on cohomology.

We let ι_n denote the standard inclusion of \mathcal{A}_n in \mathcal{S}_n .

The almost-Hopf ring structures developed in the previous section are on the whole compatible with these maps.

Proposition 3.4. *Restriction maps $res : H^*(BS_n) \rightarrow H^*(BA_n)$ preserve coproducts, and transfer maps $tr : H^*(BA_n) \rightarrow H^*(BS_n)$ preserve transfer products.*

Proof. Consider the commuting square of covering maps

$$\begin{array}{ccc} BA_i \times BA_j & \xrightarrow{Be_{i,j}} & BA_{i+j} \\ \downarrow B\iota_i \times \iota_j & & \downarrow B\iota_{i+j} \\ BS_i \times BS_j & \xrightarrow{Be_{i,j}} & BS_{i+j}. \end{array}$$

That natural maps in cohomology commute for squares of spaces gives the first result, and that transfer maps in a square of covering maps commute gives the second result. \square

Remark 3.5. This diagram does not define a pull-back of covering spaces. Instead, the space $BA_i \times BA_j$ is a double cover of the pullback, reflecting the fact that $\mathcal{A}_i \times \mathcal{A}_j$ is of index two in $\mathcal{A}_{i+j} \cap \mathcal{S}_i \times \mathcal{S}_j$, which we use in Proposition 3.14 to prove the vanishing of transfer products of classes restricted from the cohomology of symmetric groups. A similar fact accounts for the failure of Hopf ring distributivity.

Transfer maps do not preserve cup products in general, though we will see from the main calculation of Theorem 8.1 that they do preserve cup products for classes “of uniform charge.”

Restriction maps of course preserve cup products, since they are defined by maps of spaces. While restriction maps do not preserve transfer products, there is some compatibility.

Proposition 3.6.

$$\text{res}(\alpha) \odot \beta = \text{res}(\alpha \odot \text{tr}(\beta)).$$

Proof. Apply restriction maps horizontally and transfer maps vertically to the following diagram of covering maps, which is a pullback.

$$\begin{array}{ccc} BS_i \times BA_j & \xleftarrow{B\iota_i \times id} & BA_i \times BA_j \\ id \times B\iota_j \downarrow & & \downarrow Be_{i,j} \\ BS_i \times BS_j & & \\ Be_{i,j} \downarrow & & \downarrow \\ BS_{i+j} & \xleftarrow{B\iota_{i+j}} & BA_{i+j} \end{array}$$

□

3.3. The Gysin sequence. The standard restriction and transfer maps between the cohomology of alternating and symmetric groups give rise to a short exact sequence whose analysis forms the backbone of our calculations. Recall for example from Section 6.6 of [2] that because $\mathcal{A}_n \subset \mathcal{S}_n$ is an index two subgroup when $n > 1$, their classifying spaces define a principal bundle $C_2 \rightarrow BA_n \rightarrow BS_n$. Since $C_2 \cong O(1)$, we can apply the Gysin sequence for the associated line bundle, which reads

$$\dots \xrightarrow{\delta} H^k(BS_n) \xrightarrow{\text{res}} H^k(BA_n) \xrightarrow{\text{tr}} H^k(BS_n) \xrightarrow{e} H^{k+1}(BS_n) \xrightarrow{\text{res}} \dots,$$

where e is the Euler class of the line bundle.

Decomposing into short exact sequences yields

$$0 \rightarrow H^*(BS_n)/e \xrightarrow{\text{res}} H^*(BA_n) \xrightarrow{\text{tr}} \text{Ann}(e) \rightarrow 0,$$

where $\text{Ann}(e)$ is the annihilator ideal.

When $n = 2$ the associated line bundle in the Gysin sequence is the tautological line bundle over $BS_2 = \mathbb{R}P^\infty$, and in the notation of Theorem 3.1 the Euler class e is $\gamma_{1,1}$, the generator of the cohomology. In general the Euler class must be the unique non-trivial class which restricts to this, $e = \gamma_{1,1} \odot 1_{n-1}$. We record how it multiplies, which is immediate from Hopf ring distributivity.

Lemma 3.7. *The product of $e = \gamma_{1,1} \odot 1_{n-1}$ with a Hopf ring monomial produces a linear combination of monomials where each occurrence of $\gamma_{1,m}^k$ is replaced by $\gamma_{1,1}^{k+1} \odot \gamma_{1,m-1}^k$.*

Theorem 3.8. *The annihilator ideal $\text{Ann}(e)$ has a basis \mathcal{G}_a of classes of scale greater than one.*

Proof. A Hopf monomial of scale greater than one has all constituent $\gamma_{\ell,m}$ -monomials have a factor of at least one $\gamma_{\ell,m}$ with $\ell > 1$. Such Hopf ring generators, and thus monomials in them, have coproduct where no terms are supported on BS_2 . Thus scale greater than one Hopf ring monomials will annihilate e by Hopf ring distributivity.

Conversely, a Hopf ring monomial of scale one will have a constituent monomial of the form $\gamma_{1,m}^k$ with $k \geq 0$. By Lemma 3.7, the product of e with a class with such constituent monomials is non-zero, as in particular the term with the greatest k will give rise to a non-zero term. \square

Graphically, skyline diagrams for classes in $\text{Ann}(e)$ have no columns comprised entirely of 1×1 blocks, as well as no “empty spaces.”

Corollary 3.9. *The annihilator ideal $\text{Ann}(e)$ is zero when $n = 4k + 2$.*

At this point, we could describe the cohomology of alternating groups \mathcal{A}_{4k+2} as quotients of the corresponding cohomology of symmetric groups by Euler classes. The \mathcal{A}_{4k} are much more interesting, and it will be straightforward to understand the cohomology of \mathcal{A}_{4k+2} from our description of the general case.

Theorem 3.10. *A representative basis \mathcal{G}_q of the quotient of the cohomology of $B\mathcal{S}_n$ by $e = \gamma_{1,1} \odot 1_{n-1}$ is given by Hopf ring monomials in which the largest power of $\gamma_{1,m}^k$ which occurs as a constituent \cdot -monomial has $m > 1$ or $k = 0$.*

Our choices of representatives satisfy $\mathcal{G}_a \subset \mathcal{G}_q$. The skyline diagrams corresponding to \mathcal{G}_q have their tallest pure 1×1 -block building of width at least two.

Proof. To show that \mathcal{G}_q spans we induct on the difference $\delta(h)$ between the largest power of $\gamma_{1,1}$ which occurs in a monomial h and the powers of $\gamma_{1,m}$ which occur in h . When $\delta(h)$ is zero or negative, a Hopf ring monomial is in \mathcal{G}_q .

Consider a Hopf ring monomial h for which $\gamma_{1,1}^k$ is the largest power of $\gamma_{1,m}$ which occurs. Let h' be defined by replacing $\gamma_{1,1}^k$ by $\gamma_{1,1}^{k-1}$ in h , or more generally replacing $\gamma_{1,1}^k \odot \gamma_{1,m}^{k-1}$, if there is such a term, by $\gamma_{1,m+1}^{k-1}$. By Lemma 3.7 the product of e and h' has h as one term, and other terms with strictly smaller δ . Inductively, h can be written as a sum of monomials in \mathcal{G}_q .

Independence of \mathcal{G}_q follows as its span does not intersect the ideal generated by e . Any product of a Hopf ring monomial m with e is either zero or produces at least one term which is not in \mathcal{G}_q , namely the one for which $\gamma_{1,1}$ is “matched” with the highest power of any $\gamma_{1,q}$. This term which has the greatest power for $\gamma_{1,1}$ as a \cdot -monomial uniquely determines m , so the product of e with any linear combination of Hopf monomials will contain such terms. \square

The bases \mathcal{G}_a and \mathcal{G}_q are readily enumerable. To our knowledge, this is the first determination of additive structure for the cohomology of alternating groups. The calculation trails such knowledge of symmetric groups by Nakaoka [14] by over fifty years, but was relatively short work using Hopf ring structure.

3.4. The standard involution and an extension of almost-Hopf ring structure. Further computations in this Gysin sequence are facilitated by a standard involution on the cohomology of \mathcal{A}_n coming from its embedding in \mathcal{S}_n as a normal subgroup.

Definition 3.11. Denote by \bar{x} the image of $x \in H^*(B\mathcal{A}_n)$ under the action of conjugation by any element of \mathcal{S}_n not in \mathcal{A}_n , or equivalently by the non-trivial deck transformation of $B\mathcal{A}_n$ as a cover of $B\mathcal{S}_n$.

The latter definition makes the following immediate.

Proposition 3.12. *Restriction and transfer from and to the cohomology of symmetric groups are invariant under the standard involution, in that $\text{res}(\bar{y}) = \text{res}(y)$ and $\text{tr}(\bar{x}) = \text{tr}(x)$.*

We call classes in the image of restriction neutral, since involution fixes them, and informally at the moment call those which have non-zero image under transfer charged. We make charge more precise in two ways later, in which case the involution will reverse charge.

To make full use of this involution, we understand its interplay with product and coproduct structures. Let $\iota(x)$ denote the involution in homomorphism notation.

Proposition 3.13. • $\overline{x \odot y} = \overline{x} \odot y$.

- $\Delta(x)$ is invariant under $\iota \otimes \iota$.
- $\Delta(\overline{x}) = (\iota \otimes id)(\Delta(x))$ (which equals $id \otimes \iota(\Delta(x))$ by the previous).

Proof. Use the standard simplicial model for the inclusion of $B(\mathcal{A}_n \times B\mathcal{A}_m)$ in $B\mathcal{A}_{n+m}$. Conjugation on $B\mathcal{A}_{n+m}$ by any elements not in \mathcal{A}_{n+m} , in particular such elements which are in $\mathcal{S}_n \times id$ or $id \times \mathcal{S}_m$, yield the standard conjugation action. This inclusion is thus equivariant up to homotopy with respect to the projection of $C_2 \times C_2$ to C_2 by quotienting by the diagonal subgroup. This equivariance yields all of the stated equalities in cohomology. \square

A consequence of Proposition 3.13 is that $(x + \overline{x}) \odot (y + \overline{y}) = 0$. By exactness of the Gysin sequence, each of these factors is in the image of the restriction map from the cohomology of symmetric groups. More generally we have the following.

Proposition 3.14. *A transfer product in the cohomology of alternating groups of two classes restricted from the cohomology of symmetric groups is zero.*

We record the following for later use.

Lemma 3.15. *Let K be a subgroup of finite groups G and H . Suppose $K \subset K'$ of even index, with K' also a subgroup of G and H . Then the composite $H^*(BG) \xrightarrow{res} H^*(BK) \xrightarrow{tr} H^*(BH)$ is zero on mod-two cohomology.*

Proof. By assumption, BK' forms an intermediate cover between BK and both BG and BH . Thus both maps in the composite $H^*(BG) \rightarrow H^*(BK) \xrightarrow{tr} H^*(BH)$ factor through the cohomology of BK to give

$$H^*(BG) \rightarrow H^*(BK') \rightarrow H^*(BK) \xrightarrow{tr} H^*(BK') \xrightarrow{tr} H^*(BH).$$

Because $K \subset K'$ of even index, the middle composite is zero on mod-two cohomology. \square

Proof of Proposition 3.14. By definition, we consider the composite

$$H^*(B(\mathcal{S}_i \times \mathcal{S}_j)) \xrightarrow{res} H^*(B(\mathcal{A}_i \times \mathcal{A}_j)) \xrightarrow{tr} H^*(B\mathcal{A}_{i+j}).$$

As noted in the proof of Proposition 3.4, $\mathcal{A}_i \times \mathcal{A}_j$ is an index two subgroup of the intersection $\mathcal{S}_i \times \mathcal{S}_j \cap \mathcal{A}_{i+j}$ in \mathcal{S}_{i+j} . Lemma 3.15 applies to give the result. \square

Corollary 3.16. $x \odot res(y) = \overline{x} \odot res(y)$.

Proposition 3.14 has significant consequences for the global structure of the cohomology of alternating groups, and in particular the inverse system it forms. For symmetric groups, one can lift classes in this inverse system by taking transfer products with cup unit classes. For alternating groups, transfer products of neutral classes with such unit classes will result in zero, and the transfer product of a charged class with a unit class yields a lift of the sum of the class and its conjugate. That is, charged classes are inherently unstable, and the stability of neutral classes is not realized by transfer product structure as it is for symmetric groups.

For cup products, the fact that the diagonal map $B\mathcal{A}_i \rightarrow B\mathcal{A}_i \times B\mathcal{A}_i$ is equivariant with respect to the involution on $B\mathcal{A}_i$ and the diagonal involution on $B\mathcal{A}_i \times B\mathcal{A}_i$ gives the following.

Proposition 3.17. $\overline{x \cdot y} = \overline{x} \cdot \overline{y}$

These results lead to a coherent extension of almost Hopf ring structure.

Definition 3.18. Define $B\mathcal{A}'_0$ to be $S^0 = \{+, -\}$, and the product $B\mathcal{A}'_0 \times B\mathcal{A}_n \rightarrow B\mathcal{A}_n$ to be the involution on $- \times B\mathcal{A}_n$, and the identity map on $+ \times B\mathcal{A}_n$.

Let 1^+ and 1^- be the corresponding generators of $H^0(B\mathcal{A}'_0)$, so that $1^- \odot x = \overline{x}$.

Let $H^*(B\mathcal{A}_\bullet) = H^*(B\mathcal{A}'_0) \oplus \bigoplus_{m \geq 1} H^*(B\mathcal{A}_{2m})$.

Proposition 3.19. *With maps as above, $H^*(BA_\bullet)$ forms an almost Hopf ring, extending the almost Hopf ring structure on $\bigoplus_{m \geq 0} H^*(BA_{2m})$.*

Proof. Proposition 3.13 implies that the transfer product with 1^- is associative and commutative. Bialgebra structure of cup product and coproduct is still immediate because the coproduct is induced by a map of spaces. Proposition 3.17 along with the fact that $1^- \cdot 1^+ = 0$ extends Hopf distributivity to apply to transfer products with 1^- . \square

Conversely, this extended almost Hopf ring structure encodes Propositions 3.13 and 3.17. We can also check compatibility with our other results. Propositions 3.4 and 3.6 extend by Proposition 3.12. The fact that classes restricted from the cohomology of symmetric groups are invariant under involution extends Proposition 3.14. The coproduct of x will now include the terms $1^- \otimes \bar{x} + \bar{x} \otimes 1^-$, making the statement of Theorem 3.21 below more uniform.

3.5. The strategy for finding Hopf ring generators and relations. Recall from Section 3.3 the sets of elements for the cohomology of symmetric groups \mathcal{G}_a and \mathcal{G}_q , which account for the cohomology of alternating groups in the Gysin sequence. The Hopf ring generators $\gamma_{\ell, 2^k}$ for $\ell \geq 2$ generate \mathcal{G}_a , which consists of classes of scale greater than one, and those along with generators $\gamma_{1, k}$ with $k \geq 2$ and unit classes for cup product generate \mathcal{G}_q .

- (1) For $\gamma_{\ell, 2^k} \in \mathcal{G}_a$ with $\ell \geq 2$ we will find $\gamma_{\ell, 2^k}^+ \in H^*(BA_n)$ whose image under transfer is $\gamma_{\ell, 2^k}$. For consistency with notation needed in Step (3), let $\gamma_{1, 2^k; 2^k}$ be the restriction of $\gamma_{1, 2^k}$.
- (2) For each $x \in \mathcal{G}_a$ we will show there exists a Hopf ring polynomial in the $\gamma_{\ell, 2^k}^+ \in H^*(BA_n)$ and $\gamma_{1, 2^k; 2^k}$ whose image under transfer is x . We call this class x^+ and define $x^- = \overline{x^+}$, which also transfers to x . Noting that $\mathcal{G}_a \subset \mathcal{G}_q$, x must restrict to $x^+ + x^-$. Denote the set of x^+ above by \mathcal{B}_+ and the set of x^- by \mathcal{B}_- , again suppressing n from notation.
- (3) Next we show $y \in \mathcal{G}_q \setminus \mathcal{G}_a$ is the transfer product of a polynomial in $\gamma_{1, k} \odot 1_{m-k}$ with $k \geq 2$ and an element of \mathcal{G}_a from a smaller alternating group. By Proposition 3.6 and Step (2), its restriction y^o will be the transfer product of a polynomial in the restrictions of $\gamma_{1, k} \odot 1_{m-k}$, which we call $\gamma_{1, k; m}$, and an element of \mathcal{B}_+ from a smaller alternating group. Denote the set of y^o by \mathcal{B}_o .
- (4) From the Gysin sequence, the union of \mathcal{B}_+ , \mathcal{B}_- and \mathcal{B}_o form an additive basis for $H^*(BA_n)$. So the $\gamma_{\ell, 2^k}^+$ and $\gamma_{1, k; m}$, along with 1^- generate $H^*(BA_\bullet)$ as an almost Hopf ring.
- (5) Finally we will turn to relations, as well as coproduct calculations as needed to apply Hopf ring distributivity. In order to detect relations we first inductively show that the elementary abelian subgroups of \mathcal{A}_n detect its cohomology.

Cup and transfer products exhibit different behavior on \mathcal{B}_+ , \mathcal{B}_- and \mathcal{B}_o . We will see that transfer products of classes of the same charge are naturally positive and of opposite charge are negative. Cup products between classes of the same charge behave mostly like corresponding cup products for symmetric groups, while cup products between classes of opposite charge will “mostly” be zero.

3.6. Coproduct of a transfer product. While we will explicitly construct \mathcal{B}_+ , \mathcal{B}_- and \mathcal{B}_o , their characterization through the Gysin sequence allows us to understand the cohomology of \mathcal{A}_n as a C_2 -representation under the conjugation. This presentation over C_2 is key to the interplay between transfer product and coproduct.

Definition 3.20. A polarized basis for a C_2 -representation is a basis $B = \{B_+, B_-, B_o\}$ where the C_2 -action interchanges B_+ and B_- and fixes B_o . The positive projection, denoted $\rho^+(x)$ by abuse omitting B from notation, is that onto the span of B_+ .

We need to consider tensor powers of $H^*(BA_\bullet)$. For V with polarized basis, any tensor power have an induced polarized basis where the neutral sub-basis is given by the tensor products of B_o and by convention the positive sub-basis is given by products where the first non-neutral vector is positive.

Theorem 3.21. *The coproduct $\Delta(\alpha \odot \beta)$ is equal to*

$$\mu_{\odot} \otimes \mu_{\odot} (\tau \circ \rho^+ (\Delta(\alpha) \otimes \Delta(\beta))),$$

where ρ^+ is defined through the polarized basis on $H^*(BA_{\bullet})^{\otimes 4}$ induced by $\mathcal{B} = \{\mathcal{B}_+, \mathcal{B}_-, \mathcal{B}_o\}$, and where τ is the standard transposition of second and third factors of the tensor product.

Recalling that μ_{\odot} is the multiplication map for the transfer product, this differs from the usual statement that (\odot, Δ) form a bialgebra only by the polarization ρ^+ . For brevity, we express the equality in Theorem 3.21 as $\Delta(\alpha \odot \beta) = \Delta\alpha \odot_{\rho^+} \Delta\beta$.

Before proving this theorem, we recall the Borel spectral sequence for the cohomology of the quotient of X by a free action of G . This is the Leray-Serre spectral sequence for the fibration $X \rightarrow X/G \rightarrow BG$, using the fact that $X/G \simeq X \times_G EG$, so $E_2^{p,q} = H^p(BG; \mathcal{H}^q(X))$.

We apply this at first to the case where C_2 acts on BA_n by conjugation, with quotient BS_n . While we already know the cohomology of BS_n , this spectral sequence will be generalized in proving Theorem 3.21.

Over $k = \mathbb{F}_2$ there are only two C_2 -modules to consider, namely the trivial module and the regular representation. The cohomology of BC_2 with trivial coefficients is that of $\mathbb{R}P^\infty$, while as usual the cohomology of the regular representation is concentrated in degree zero, of rank one. As every conjugate pair $x^\pm \in \mathcal{B}_\pm$ gives a copy of the regular representation, they give rise to a single class which we call $x \in E_{0,*}^2$ which then restricts to $x^+ + x^-$, consistent with previous definitions of these classes. For every $y \in \mathcal{B}_o$ we have a corresponding $y \in E_{0,*}^2$ and more generally $y \cdot e^q \in E_2^{q,*}$. Straightforward calculation using Theorems 3.1, 3.8 and 3.10 shows that these x and $y \cdot e^q$ on the E^2 -page give spaces of the same rank as the cohomology of symmetric groups, so the spectral sequence collapses at E^2 .

Proof of Theorem 3.21. Recalling the proof of Theorem 2.5, let $p + q = n$, $r + s = m$, $p + r = i$, and $q + s = j$, and set $d = i + j = n + m$. Let $H_{p,q,r,s}$ be the intersection of $\mathcal{A}_n \times \mathcal{A}_m$ and the conjugate of $\mathcal{A}_i \times \mathcal{A}_j$ in \mathcal{A}_d which contains $\mathcal{A}_p \times \mathcal{A}_q \times \mathcal{A}_r \times \mathcal{A}_s$ as a subgroup of index two.

Consider the diagram

$$\begin{array}{ccc} \sqcup BA_p \times BA_q \times BA_r \times BA_s & & \\ \downarrow f = \sqcup f_{p,q,r,s} & \searrow \sqcup \tau \circ (B \iota_{p,r} \times \iota_{q,s}) & \\ \sqcup BH_{p,q,r,s} & \xrightarrow{g = \sqcup g_{p,q,r,s}} & BA_i \times BA_j \\ \downarrow h & & \downarrow B\iota_{i,j} \\ BA_n \times BA_m & \xrightarrow{B\iota_{n,m}} & BA_{i+j}, \end{array}$$

where all maps on classifying spaces are induced by inclusions of subgroups, with some already named inclusions indicated. By Proposition 2.4, the lower square is a pull back so the coproduct of a \odot -product, which is given by $B\iota_{n,m}^* \circ B\iota_{i,j}^!$, is equal to $h^! \circ g^*$.

The failure of (\odot, Δ) to be a bialgebra is given by the existence of the $f_{p,q,r,s}$, so their analysis plays a key role. As $f_{p,q,r,s}$ is a double cover and we understand the cohomology of $BA_p \times BA_q \times BA_r \times BA_s$ as a module over $k[C_2]$, we use the Borel spectral sequence.

There is a class in $E_2^{0,*}$ for every pair $x_1 \otimes x_2 \otimes x_3 \otimes x_4$ and $\overline{x_1} \otimes \overline{x_2} \otimes \overline{x_3} \otimes \overline{x_4}$ in $\mathcal{B}^{\otimes 4} \setminus \mathcal{B}_o^{\otimes 4}$. (Here and elsewhere we use notation for the cohomology of different \mathcal{A}_n as if they were all the same vector space.) The other classes in $E_2^{p,q}$ are given by $(y \cdot e^q)$ where $y \in \mathcal{B}_o^{\otimes 4}$ and $e \in E_2^{1,0}$ is a generator for the cohomology of BC_2 .

We claim that this second set of classes are associated graded of classes which map to zero under $h^!$ (that is, choices for these classes with all possible indeterminacies map to zero under $h^!$). At the spectral

sequence level they are pulled back from the Borel spectral sequence for

$$B(\mathcal{A}_p \times \mathcal{A}_q \times \mathcal{A}_r \times \mathcal{A}_s) \rightarrow B(\mathcal{S}_p \times \mathcal{S}_q \times \mathcal{S}_r \times \mathcal{S}_s) \rightarrow B(C_2 \times C_2 \times C_2 \times C_2),$$

where the map of base spaces is induced by the diagonal embedding $C_2 \rightarrow (C_2)^4$. Thus these classes are associated graded classes pulled back from $B(\mathcal{S}_p \times \mathcal{S}_q \times \mathcal{S}_r \times \mathcal{S}_s)$. Apply Lemma 3.15 to the inclusions of $H_{p,q,r,s}$ in both $\mathcal{S}_p \times \mathcal{S}_q \times \mathcal{S}_r \times \mathcal{S}_s$ and $\mathcal{A}_n \times \mathcal{A}_m$, whose intersection in $\mathcal{S}_n \times \mathcal{S}_m$ contains $H_{p,q,r,s}$ with even index, to show that classes in $H_{p,q,r,s}$ pulled back from $B(\mathcal{S}_p \times \mathcal{S}_q \times \mathcal{S}_r \times \mathcal{S}_s)$ map to zero under $h^!$.

To prove the theorem, consider $\alpha = (\tau \circ (\Delta_{p,r} \otimes \Delta_{q,s}))(x \otimes y) \in H^*(B\mathcal{A}_p \times B\mathcal{A}_q \times B\mathcal{A}_r \times B\mathcal{A}_s)$. As in the Gysin sequence for alternating and symmetric groups, its polarization will transfer under $f^!$ to a class β with $f_{p,q,r,s}^*(\beta) = \alpha$ modulo $\mathcal{B}_o^{\otimes 4}$. Thus by the analysis above, its polarization transfers under $f^!$ to something whose difference from $g_{p,q,r,s}^*(x \otimes y)$ is in the kernel of $h^!$. Therefore $h^! \circ g_{p,q,r,s}^*(x \otimes y)$ is equal to $(h \circ f_{p,q,r,s})^!$ applied to the polarization of $\Delta_{p,r} \times \Delta_{q,s}(x \otimes y)$, as claimed. \square

4. FOX-NEUWIRTH MODELS

At key points, starting with the definition of our almost Hopf ring generators, we require cochain-level calculations. Rather than the cobar construction for group rings, we prefer cochain models based on the geometry of configuration spaces, due to Fox and Neuwirth. We first briefly recall these for symmetric groups, as developed in [9].

We choose the classifying space for the symmetric group \mathcal{S}_n as the space of distinct points in \mathbb{R}^∞ , which we call $\overline{\text{Conf}}_n(\mathbb{R}^\infty)$, which is the quotient of the labeled configuration space $\text{Conf}_n(\mathbb{R}^\infty)$ by the symmetric group action permuting labels. The finite-dimensional approximations $\overline{\text{Conf}}_n(\mathbb{R}^d)$ are manifolds with a beautiful cellular decomposition.

The points in any configuration in $\overline{\text{Conf}}_n(\mathbb{R}^d)$ are ordered by the dictionary order of their coordinates. If we consider the i th and $i + 1$ st points under this ordering, they share some a_i of their first coordinates. That is, $a_i = 0$ if their first coordinates are distinct, $a_i = 1$ if they share their first coordinate but have distinct second coordinates, and so forth.

Definition 4.1. Let $\Gamma = [a_1, \dots, a_{n-1}]$ be a sequence of non-negative integers, and let $|\Gamma| = \sum a_i$. Define $\overline{\text{Conf}}_\Gamma(\mathbb{R}^d)$ to be the collection of all configurations such that the i th and $i + 1$ st points in the dictionary order in the configuration share their first a_i coordinates but not their $(a_i + 1)$ st. We say such points respect Γ .

Theorem 4.2 (after Fox-Neuwirth). *For any Γ the subspace $\overline{\text{Conf}}_\Gamma(\mathbb{R}^d)$ is homeomorphic to a Euclidean ball of dimension $nd - |\Gamma|$. The images of the $\overline{\text{Conf}}_\Gamma(\mathbb{R}^d)$ are the interiors of cells in a CW structure on the one-point compactification $\overline{\text{Conf}}_n(\mathbb{R}^d)^+$.*

Definition 4.3. Let $(FN_n^d)_*$ be the cellular chain complex associated to the cell structure defined by the $\overline{\text{Conf}}_\Gamma(\mathbb{R}^d)$, which by the above computes the homology of $\overline{\text{Conf}}_n(\mathbb{R}^d)^+$.

These chain complexes model the cochains of the classifying spaces of symmetric groups as follows. Assume that d is even, in which case $\overline{\text{Conf}}_n(\mathbb{R}^d)$ is an orientable manifold of dimension nd . Alexander Duality implies that the homology of its one-point compactification in degree $nd - i$ is isomorphic to its cohomology in degree i . If $i < d$, the group $(FN_n^d)_{nd-i}$ and differential is independent of d , so we may set the following.

Definition 4.4. Let FN_n^* be the cochain complex which in degree $* = i$ is $(FN_n^d)_{nd-i}$ for $d > i$ and with differential defined through the cellular chain structure on $(FN_n^d)_*$.

We show in [8] that the cohomology of FN_n^* is that of $B\mathcal{S}_n$. While the chain groups $(FN_n^d)^i$ are simple, spanned by sequences of non-negative integers which add up to i , the boundary maps are complicated. A main result of [9] is explicit calculation of the differential.

We develop the alternating group analogues presently. For alternating groups, our cochains will be a double cover of the cochains for symmetric groups, and we use orientation or “charge” to express this.

Definition 4.5. A charged sequence of non-negative integers is a finite sequence of non-negative integers along with a choice of sign. Given a non-negative sequence of integers Γ we write Γ^+ for the positively charged sequence associated to Γ and Γ^- for the negatively charged sequence. For convenience, we write $\Gamma^\circ = \Gamma^+ + \Gamma^-$ in chain groups.

Let FNA_n^i be spanned by charged sequences of $(n-1)$ non-negative integers which sum to i .

Identifying the differential involves a few combinatorial notions.

Definition 4.6. The ℓ -blocks of a sequence $\Gamma = [a_1, \dots, a_{n-1}]$ are the ordered collection of possibly empty subsequences $[a_i, a_{i+1}, \dots, a_{i+k}] \subset \Gamma$ such that a_{i-1} and a_{i+k+1} are consecutive in the subset of entries which are less than or equal to ℓ . By convention, set $a_0 = a_n = -\infty$, so in particular an empty sequence always has a single empty ℓ -block for any ℓ .

Denote by $\Gamma\langle i \rangle$ the sequence obtained from Γ by adding one to its i th entry.

Sometimes we call a zero-block simply a block. For example, the blocks of $\Gamma_{\text{ex}} = [3, 0, 1, 2, 0, 0, 4, 4]$ are $([3], [1, 2], \emptyset, [4, 4])$, while $\Gamma_{\text{ex}}\langle 2 \rangle$ has one-blocks $([3], \emptyset, [2], \emptyset, [4, 4])$.

In the cell $\overline{\text{Conf}}_\Gamma(\mathbb{R}^d)$, ℓ -blocks of length k correspond to collections of $k+1$ adjacent points in a configuration which share more than their first ℓ coordinates.

Definition 4.7. Let $\Gamma = [a_1, \dots, a_{n-1}]$ and fix $1 \leq i \leq n-1$. The sequence $\Gamma\langle i \rangle$ has a non-empty a_i -block of the form $\Lambda_i = [a_{i-r}, \dots, a_i + 1, \dots, a_{i+s}]$ for some $r, s \geq 0$.

Partition the $(a_i + 1)$ -blocks of Λ_i into the $p > 0$ of them appearing in the possibly empty subsequence $[a_{i-r}, \dots, a_{i-1}]$ and the $q > 0$ of them in $[a_{i+1}, \dots, a_{i+s}]$. Write $\text{Sh}(\Gamma, i)$ for the collection of (p, q) -shuffles with action on $\Gamma\langle i \rangle$ by permuting the $a_i + 1$ blocks of Λ_i and leaving the remainder of the sequence fixed.

Denote by $\text{Sh}_+(\Gamma, i)$ and $\text{Sh}_-(\Gamma, i)$ the subsets of even and odd shuffles in $\text{Sh}(\Gamma, i)$, respectively.

Continuing with Γ_{ex} , we find $\Lambda_2 = [3, 1, 1, 2]$ has 1-blocks $([3], \emptyset, [2])$, partitioned into $([3])$ and $(\emptyset, [2])$. The set $\text{Sh}(\Gamma_{\text{ex}}, 2)$ consists of the three $(1, 2)$ -shuffles, which act on $\Gamma_{\text{ex}}\langle 2 \rangle$ by sending it to $[3, 1, 1, 2, 0, 0, 4, 4]$, $[1, 3, 1, 2, 0, 0, 4, 4]$, and $[1, 2, 1, 3, 0, 0, 4, 4]$.

Definition 4.8. Let Γ be a sequence of non-negative integers. The mod-two differential in FNA_n^* is given by

$$\delta(\Gamma^\pm) = \sum_{i=1}^{n-1} \delta_i(\Gamma^\pm) \quad \text{where} \quad \delta_i(\Gamma^\pm) = \sum_{\sigma \in \text{Sh}_+(\Gamma, i)} \sigma \cdot \Gamma\langle i \rangle^\pm + \sum_{\sigma \in \text{Sh}_-(\Gamma, i)} \sigma \cdot \Gamma\langle i \rangle^\mp$$

This gives our cochain model for classifying spaces of alternating groups.

Theorem 4.9. *The cohomology of FNA_n^* is that of $B\mathcal{A}_n$.*

We start proving this theorem by developing preferred classifying spaces.

Definition 4.10. Let $\text{Ind}_n(\mathbb{R}^d)$ be the subset of $\text{Conf}_n(\mathbb{R}^d)$ of configurations which are linearly independent. The span of such configurations has a canonical orientation, coming from the configuration as an ordered basis.

Let $\overline{\text{OrInd}}_n(\mathbb{R}^d)$ be the quotient of $\text{Ind}_n(\mathbb{R}^d)$ by the alternating group action on the labels, which thus preserves the canonical orientation.

Thus $\overline{\text{OrInd}}_n(\mathbb{R}^d)$ is the space of unlabeled configurations with an orientation on their span. Because $\text{Ind}_n(\mathbb{R}^d)$ is $(d-n-1)$ -connected, $\overline{\text{OrInd}}_n(\mathbb{R}^\infty)$ models $B\mathcal{A}_n$.

Definition 4.11. Let $\Gamma = [a_1, \dots, a_{n-1}]$ be a sequence of non-negative integers. Define $\overline{\text{OrInd}}_{\Gamma^+}(\mathbb{R}^d)$ (respectively $\overline{\text{OrInd}}_{\Gamma^-}(\mathbb{R}^d)$) to be the collection of all configurations with the following two properties

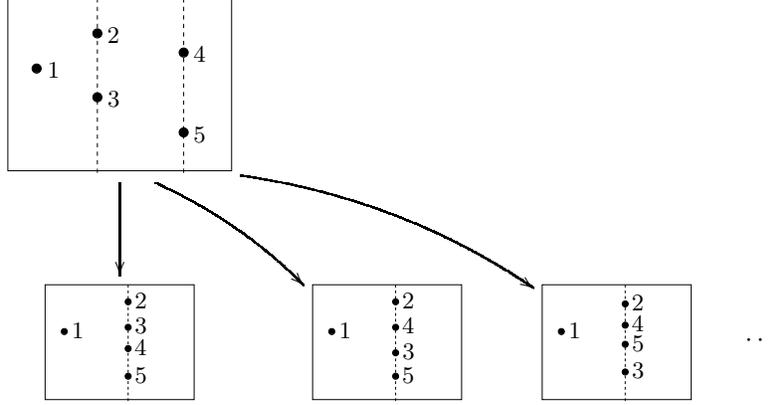


FIGURE 1. The boundary of a “cell” in $\overline{\text{OrInd}}_{\Gamma_{\pm}}(\mathbb{R}^d)$ involves shuffles of ordered families of points.

- the i th and $i + 1$ st points in the dictionary order in the configuration share their first a_i coordinates but not their $a_i + 1$ st;
- the orientation of the span defined by the dictionary order agrees (respectively, does not agree) with the orientation of the span as a point in $\overline{\text{OrInd}}_n(\mathbb{R}^d)$.

Proof of Theorem 4.9. The subspace $\overline{\text{OrInd}}_{\Gamma_{\pm}}(\mathbb{R}^d)$ is not the interior of a cell, but is the complement within a cell of the subvariety of non-linearly independent configurations, which is of codimension $d - n + 1$. Taking closures in the one-point compactification, the long exact sequences of pairs of these codimension i (not-quite-)“cells” will behave as if they were cellular in and around degree i as long as $d > i + n$. Moreover, the set of “cells” and their boundary behavior will be independent of d as long as $d > i$.

So as long as $d > i + n$, the i th cohomology of $\overline{\text{OrInd}}_n(\mathbb{R}^d)$ agrees with that of BA_n and is computed by the incidence of the $\overline{\text{OrInd}}_{\Gamma_{\pm}}(\mathbb{R}^d)$ as if they were cellular. The spectral sequences associated to filtration by $|\Gamma|$ form directed system which stabilizes and thus converges in the limit. The stable terms yield the chain groups FNA_n^i as the $E_0^{i,0}$.

For the boundary maps, observe that each “cell” is bounded by a family of such for which some pair of points consecutive in the dictionary ordering agree in one more dimension. Suppose that the i th and $(i + 1)$ st points in the “cell” share their first a_i coordinates. We know that the i th point has a smaller $(a_i + 1)$ st coordinate than the $(i + 1)$ st point, but we do not know the relationship between their $(a_i + 2)$ nd coordinates. Further, it is possible for either of these points to share $(a_i + 1)$ coordinates with other points, in which case the ordering on this whole collection of points in the boundary is only partially determined by the data in the “cell”. See, for example, Figure 1. Thus, every shuffle of these two ordered families of points appears in the boundary of the “cell”, with the orientation of the bounding “cell” changing under odd shuffles. \square

The two-sheeted covering of BA_n over BS_n has the effect of splitting the cells which correspond to cycles in FN_n^* . For example, in FN_4^3 , we have

$$\begin{aligned} \delta([1, 0, 2]) &= 2[2, 0, 2] + [1, 1, 2] + [2, 1, 2] + [2, 1, 1] \\ \delta([2, 0, 1]) &= 2[2, 0, 2] + [1, 1, 2] + [2, 1, 2] + [2, 1, 1] \end{aligned}$$

so their sum is a mod-two cycle which by the symmetric group analogue of Theorem 4.17 below represents $\gamma_{1,1}^2 \odot \gamma_{1,1}$. While in FNA_4^3 ,

$$\begin{aligned} \delta([2, 0, 1]^\pm) &= [3, 0, 1]^\pm + [3, 0, 1]^\mp + [2, 1, 1]^\pm + [1, 2, 1]^\pm + [1, 1, 2]^\pm + [2, 0, 2]^\pm + [2, 0, 2]^\mp \\ &= [3, 0, 1]^o + [2, 1, 1]^\pm + [1, 2, 1]^\pm + [1, 1, 2]^\pm + [2, 0, 2]^o \end{aligned}$$

Thus $[2, 0, 1]^o + [1, 0, 2]^o$ is a cycle, pulled back from the cycle $[2, 0, 1] + [1, 0, 2] \in \text{FN}_4^3$. However it is trivial in cohomology as, for example,

$$\delta([1, 0, 1]^\pm) = [2, 0, 1]^o + 4[1, 1, 1]^\pm + 2[1, 1, 1]^\mp + [1, 0, 2]^o.$$

The distribution of even and odd permutations in a cell's boundary is generally not symmetric. Thus we require computations of $|\text{Sh}_\pm(p, q)|$. The results are elementary and will be stated without proofs, which can use either bijective arguments or the basic fact that $|\text{Sh}_\pm(p, q)| = |\text{Sh}_\pm(p-1, q)| + |\text{Sh}_{\pm(-1)^p}(p, q-1)|$. The following are some of the key computations using such results.

Lemma 4.12. *Suppose $\Gamma^\pm = [a_1, \dots, a_{n-1}]^\pm$ and $[a_{i-r}, \dots, a_{i-1}]$ and $[a_{i+1}, \dots, a_{i+s}]$ are a_i -blocks of Γ . If $a_j = a_i + 1$ for every $j \in \{i-r, \dots, i-1, i+1, \dots, i+s\}$, then*

$$\delta_i(\Gamma^\pm) = |\text{Sh}_+(\Gamma, i)|\Gamma\langle i \rangle^\pm + |\text{Sh}_-(\Gamma, i)|\Gamma\langle i \rangle^\mp.$$

In particular,

$$\delta_i(\Gamma^\pm) = \begin{cases} \Gamma^o\langle i \rangle, & \text{if } r = s = 0 \\ 0, & \text{if } r = s > 0 \\ \Gamma\langle i \rangle^\pm, & \text{if } \ell > 1, r = 2^\ell - 1, \text{ and } s < r. \end{cases}$$

Lemma 4.13. *Let Γ^\pm be as in Lemma 4.12.*

If $a_j > a_i + 1$ for all $j \in \{i-r, \dots, i-1\}$, and $a_j = a_i + 1$ for $j \in \{i+1, \dots, i+s\}$, with $r > 1$ and $s > 0$, then

$$\delta_i(\Gamma^\pm) = \sum_{j=0}^{r+1} [a_1, \dots, a_{i-r-1}, \hat{a}_{(i-r,j)}, \dots, \hat{a}_{(i+1,j)}, a_{i+s+1}, \dots, a_{n-1}]^{\pm(-1)^j},$$

where $\hat{a}_{(k,j)} = [a_{i-r}, \dots, a_{i-1}]$ if $k = i-r+j$ and $\hat{a}_{(k,j)} = a_i + 1$ otherwise.

For example,

$$\begin{aligned} \delta_5([1, 0, 2, 3, 0, 1, 1, 0, 1]^o) &= [2, 0, 2, 3, 1, 1, 1, 0, 1]^o + [2, 0, 1, 2, 3, 1, 1, 0, 1]^o \\ &\quad + [2, 0, 1, 1, 2, 3, 1, 0, 1]^o + [2, 0, 1, 1, 1, 2, 3, 0, 1]^o. \end{aligned}$$

The proofs are straightforward calculation. For Lemma 4.12, Λ_i has $r+s+2$ empty (a_i+1) -blocks, and $\text{Sh}(\Gamma, i) = \text{Sh}(r+1, s+1)$, each of which fixes the sequence $\Gamma\langle i \rangle$. For Lemma 4.13, $\text{Sh}(\Gamma, i) = \text{Sh}(r+1, 1)$, with action on Λ_i resulting in all possible placements of the sequence $[a_{i+1}, \dots, a_{i+s}]$ in a sequence of $r+1$ (a_i+1) s with alternating charges.

Turning to more general results, we have the following.

Theorem 4.14. *Under the isomorphisms of Theorems 4.2 and 4.9, the restriction map sends Γ to $\Gamma^+ + \Gamma^-$ and the transfer map sends Γ^\pm to Γ .*

We will see in the next section that at the level of cohomology there does not seem description of the Gysin sequence through adding or dropping labels alone, especially in the cohomology of \mathcal{A}_4 . Most Hopf ring generators do behave as the cochains do in Theorem 4.14, but the \mathcal{A}_4 behavior propagates throughout.

Proof. We may use $\text{Ind}_n(\mathbb{R}^\infty)$, which is a subspace of our usual $\text{Conf}_n(\mathbb{R}^\infty)$, as a model for ES_n . The analysis of finite-dimensional approximations proceeds as in the proof of Theorem 4.9, with the filtration by $\text{Ind}_\Gamma(\mathbb{R}^d)/\mathcal{S}_n$ yielding a chain complex as the limit of associated spectral sequences. The limiting cochain complex is exactly FN_n^* . With this model for the classifying spaces of symmetric groups, the

restriction and transfer maps are “cellular” - that is, they are induced by filtration preserving maps which thus produce maps on limiting chain complexes - and are as stated. \square

Finally, we evaluate Fox-Neuwirth cochains directly by realizing duality through elementary chain-level intersection theory as developed in the class notes [17]. The theory is being extended to give cup- i structures and conjecturally E_∞ models for cochains of manifolds in [7].

Briefly, let X be a manifold and W a codimension- d manifold with an immersion i to X . We say that a smooth chain is transverse to an immersion when it is transverse in the usual sense when restricted to every pair of a face (including the interior of the simplex) and a codimension one subspace as a manifold with boundary. Define the function τ_W on a smooth chain $\sigma : \Delta^d \rightarrow X$ transverse to i as the cardinality mod-two of the pull-back of i and σ . When i and σ are embeddings, this counts intersection mod-two.

Chains transverse to a fixed, finite set of immersions forms a subcomplex quasi-isomorphic to the singular chain complex. We view τ_W as an element of the dual cochains, which while being “partially defined” can be used for many applications in cohomology.

Standard constructions in homology and cohomology theory are geometric in this model. For example, if $f : Y \rightarrow X$ is transverse to i , then the natural map on cochains is given by $f^\# \tau_W = \tau_{f^{-1}W}$. The Künneth map is given by geometric product. Coboundary is given by a Stokes formula $\delta \tau_W = \tau_{\partial W}$, which essentially follows classification of one-manifolds applied to $\sigma^{-1}(W)$ where $\sigma : \Delta^{d+1} \rightarrow X$. Poincaré duality is almost a tautology, as we can allow for W to have corners, in which for example case a triangulation of X gives rise to cochains as well as chains.

Proposition 4.15. *The isomorphism of Theorem 4.9 is realized by sending the chain Γ^\pm to the (partially defined) cochain $\tau_{\overline{\text{OrInd}}_{\Gamma^+}(\mathbb{R}^d)}$.*

Proof. In our setting, the inclusions of the $\overline{\text{OrInd}}_{\Gamma^+}(\mathbb{R}^d)$ extend to proper immersions. The Stokes formula then shows that these cochains form a subcomplex of the transverse cochain complex. By Theorem 4.9, this subcomplex computes the appropriate cohomology. \square

Theorem 4.16. *The coproduct Δ is cellular in FNA_*^* , given by sending*

$$[a_1, \dots, a_n]^\pm \mapsto \sum [a_1, \dots, a_{i-1}]^+ \otimes [a_{i+1}, \dots, a_n]^\pm + [a_1, \dots, a_{i-1}]^- \otimes [a_{i+1}, \dots, a_n]^\mp,$$

where the sum is over all i such that $a_i = 0$.

Proof. We use the fact that for $f : X \rightarrow Y$ transverse to W we have $f^\# \tau_W = \tau_{f^{-1}W}$. Consider the cochain τ_W where W is the immersion of the cell labelled by $\Gamma = [a_1, \dots, a_n]$, and let $Y = \overline{\text{OrInd}}_n(\mathbb{R}^d) \times \overline{\text{OrInd}}_m(\mathbb{R}^d)$, $X = \overline{\text{OrInd}}_{n+m}(\mathbb{R}^d)$, and f be defined by “stacking” configurations. More explicitly, define f using homeomorphisms of \mathbb{R} with $(0, 1)$ and $(2, 3)$ to produce a configuration of $n + m$ points from two given configurations by taking a union of n points whose first coordinate is in $(0, 1)$ and m points whose first coordinate is in $(2, 3)$. To guarantee linear independence, we may fix modifications of coordinates beyond $|\Gamma|$.

Since f is an inclusion of a codimension zero manifold, it is transverse to any submanifold. The image of f only contains points whose n th and $n + 1$ st points differ in first coordinate, so the pullback of cochains associated to Γ with $a_n \neq 0$ will be zero. For Γ with $a_n = 0$ the preimage of $\overline{\text{OrInd}}_{\Gamma^+}(\mathbb{R}^d)$ in Y will be the union of $\overline{\text{OrInd}}_{[a_1, \dots, a_{n-1}]^\pm}(\mathbb{R}^d) \times \overline{\text{OrInd}}_{[a_{n+1}, \dots, a_{n+m-1}]^\pm}(\mathbb{R}^d)$. As the Künneth map is given by product of submanifolds, we obtain the result. The statement for $\overline{\text{OrInd}}_{\Gamma^-}(\mathbb{R}^d)$ is similar. \square

Theorem 4.17. *The transfer product is modeled at the Fox-Neuwirth cochain level by sending $\Gamma^\pm \otimes \Lambda^\pm$ to the sum over sequences whose 0-blocks are shuffles of the 0-blocks of Γ and Λ , with charge which is the product of their charges.*

Proof. We model the transfer product using the maps

$$\overline{\text{OrInd}}_n(\mathbb{R}^d) \times \overline{\text{OrInd}}_m(\mathbb{R}^d) \xleftarrow{p_1 \times p_2} \overline{\text{OrInd}}_{n,m}(\mathbb{R}^d) \xrightarrow{\phi} \overline{\text{OrInd}}_{n+m}(\mathbb{R}^d),$$

where $\overline{\text{OrInd}}_{n,m}(\mathbb{R}^d)$ is the space of configurations which are bi-colored with n points of the first color and m points of the second, each collection oriented; where p_i projects onto the i th colored subset; and where ϕ forgets colors altogether and takes the direct sum orientation. As d goes to ∞ , the product $p_1 \times p_2$ models the equivalence $B(\mathcal{A}_n \times \mathcal{A}_m) \simeq B\mathcal{A}_n \times B\mathcal{A}_m$ and ϕ is a covering map model for $B(\mathcal{A}_n \times \mathcal{A}_m) \rightarrow B\mathcal{A}_{n+m}$.

Take a chain σ on $\overline{\text{OrInd}}_{n+m}(\mathbb{R}^d)$ which is transverse to all Fox-Neuwirth cells. The transfer of $\Gamma^\pm \otimes \Lambda^\pm$ is defined by evaluating Γ^\pm and Λ^\pm on $p_1(\tilde{\sigma})$ and $p_2(\tilde{\sigma})$ as $\tilde{\sigma}$ ranges over lifts of σ , which correspond to bi-colorings of underlying configurations along with compatible pairs of orientations. Under our transversality assumption on σ , the configurations in $p_1(\tilde{\sigma})$ and $p_2(\tilde{\sigma})$ will respect Γ and Λ if and only if those in σ satisfy some sequence which is a shuffle of the zero-blocks Γ and Λ . Moreover, the orientations can then be chosen compatibly if and only if the orientation is the product of orientations, which establishes the result. \square

We obtain the following refinement of Proposition 3.14, as the transfer product of cochains which are invariant under conjugation will cancel in pairs.

Corollary 4.18. *The transfer product of two Fox-Neuwirth cochains which each are invariant under conjugation is zero.*

While Theorems 4.16 and 4.17 establish that Fox-Neuwirth cochains can be used to calculate coproducts and transfer products, they cannot be used for cup products, as claimed incorrectly in [9]. For this reason, we bring in knowledge of cup coproduct on the Cohen-Lada-May approach to homology for two cases of finding cochain representatives for cup products in Appendix A.

5. RESTRICTION TO ELEMENTARY ABELIAN SUBGROUPS

Restriction to elementary abelian subgroups is a primary tool in group cohomology, giving maps to classical rings of invariants. See for example [2], where starting in the third chapter such techniques are developed and used extensively.

For symmetric groups, the starting point is rings of invariants of polynomial algebras by general linear groups over finite fields, which were first studied by Dickson (see for example Chapter 6 of [2]). One must take these Dickson algebras, which remarkably are polynomial, and look at symmetric invariants of tensor powers. Such symmetric polynomials in multiple variables are complicated. Previous investigators of cohomology of symmetric groups proceeded by computing these invariants, with Feshbach [5] being the most successful.

While our Hopf ring approach to symmetric groups in [8] did not proceed through such invariant theory, for reference we did connect with that approach. We make use of that connection here.

Definition 5.1. For $n > 1$ let $V_n^+ \cong (C_2)^n$ denote the subgroup of \mathcal{A}_{2^n} defined by having $(C_2)^n$ act on itself. Let V_n^- be the conjugate of V_n^+ by any element of \mathcal{S}_{2^n} not in \mathcal{A}_{2^n} .

The low-dimensional cases are exceptional. As \mathcal{A}_2 is trivial, V_1^\pm must be as well. The case of \mathcal{A}_4 is exceptional in that there is only one elementary abelian 2-subgroup, so $V_2^+ = V_2^-$, which we just call V_2 . The calculation, which is worked out in detail in the first section of Chapter III of [2], significantly complicates the cohomology of all alternating groups.

The Weyl group for V_2 in \mathcal{A}_4 is not $GL_2(\mathbb{F}_2)$, which has order 6, but is instead a cyclic group of order 3. Indeed V_2 is normal in \mathcal{A}_4 , and the quotient has order three. The invariant theory approach to cohomology is quite effective.

Theorem 5.2. $H^*(B\mathcal{A}_4) \cong H^*(V_2)^{C_3} \cong \mathbb{F}_2[x_1, x_2]^{C_3}$, where C_3 acts by cyclicly permuting x_1, x_2 and $x_1 + x_2$. This ring of invariants has a generator in degree two, namely $a = x_1^2 + x_1x_2 + x_2^2$, and two in

degree three, namely $b_+ = x_1^3 + x_1^2x_2 + x_2^3$ and $b_- = x_1^3 + x_1x_2^2 + x_2^3$. There is a relation, namely $b_+^2 + b_+b_- + b_-^2 + a^3 = 0$.

This relation in the ring of invariants stands in contrast to the Dickson invariant setting, and will propagate throughout the cohomology of alternating groups.

We realize the generators of this cohomology as Fox-Neuwirth cochains.

Definition 5.3. • Let $\gamma_{1,2;2}$ be the class represented by $[1, 0, 1]^o$.

- Let $\gamma_{2,1}^+$ to be represented by $[1, 1, 1]^+ + [2, 0, 1]^o$.
- Let $\gamma_{2,1}^-$ to be represented by $[1, 1, 1]^- + [2, 0, 1]^o$.

By Theorem 4.14 and Theorem 4.9 of [8], $\gamma_{1,2;2}$ is the restriction of $\gamma_{1,2} \in H^2(B\mathcal{S}_4)$. Thus by Theorem 7.8 of [8], it restricts to a in the cohomology V_2 .

That $\gamma_{2,1}^\pm$ are cocycles is straightforward. Again using Theorem 4.14 and Theorem 4.9 of [8] we see that $\gamma_{2,1}^\pm$ both transfer to $\gamma_{2,1}$, which restricts to their sum. Thus they span $H^3(B\mathcal{A}_4)$. Because they are non-trivially conjugated, they must restrict to b_+ and b_- . Summarizing we have the following.

Proposition 5.4. *The cohomology of \mathcal{A}_4 is generated by $\gamma_{1,2;2}$, $\gamma_{2,1}^+$ and $\gamma_{2,1}^-$ with the relation*

$$\gamma_{2,1}^+ \cdot \gamma_{2,1}^- = (\gamma_{2,1}^+)^2 + (\gamma_{2,1}^-)^2 + (\gamma_{1,2;2})^3.$$

In contrast to this relation, we will see that Hopf ring generators defined on \mathcal{A}_{2^n} for $n > 2$ have mixed-charge products which are zero. Ironically, the Gysin sequence for \mathcal{A}_4 , and thus its contributions to the Gysin sequence for all alternating groups, is relatively complicated. For concreteness we record the following.

Proposition 5.5. *A class which maps to $\gamma_{1,2;2^p}\gamma_{2,1}^n \in \mathcal{B}_a$ under transfer is $\gamma_{1,2;2^p} \left(\sum_{i < \frac{n}{2}} \binom{n}{i} \gamma_{2,1}^{+i} \gamma_{2,1}^{-n-i} \right)$.*

Proof. As mentioned above, while V_2 is a subgroup of both \mathcal{A}_4 and \mathcal{S}_4 its Weyl group is different in each case. The invariants in the \mathcal{S}_4 setting sit inside those in the \mathcal{A}_4 setting, and this inclusion of invariants represents the restriction homomorphism, as the Euler class $\gamma_{1,1}$ also generates kernel of the restriction from \mathcal{S}_4 to V_2 . The transfer map takes the sum of a class and its conjugate, which on the cohomology of V_2 is represented by interchanging x_1 and x_2 .

Given this model for the transfer map, the binomial theorem implies that $\sum_{i < \frac{n}{2}} \binom{n}{i} \gamma_{2,1}^{+i} \gamma_{2,1}^{-n-i}$ transfers to $\gamma_{2,1}^n$. The fact that symmetrization of a product $f \cdot g$ when g is already symmetric is the product of the symmetrization of f with g gives the result in general. \square

Remark 5.6. The maps in the Gysin sequence on these generators are given by adding or removing labels, but unlike for Hopf ring generators of greater levels this is not the case for multiples of generators. Thus it seems that there is no basis for $H^d(B\mathcal{A}_4)$ which is readily compatible with the Gysin sequence for $d \geq 9$. For example, when $d = 9$ the restriction of $\gamma_{2,1}^3$ is $\gamma_{2,1}^{+3} + \gamma_{2,1}^{+2}\gamma_{2,1}^- + \gamma_{2,1}^+\gamma_{2,1}^{-2} + \gamma_{2,1}^{-3}$ while that of $\gamma_{1,2;2^3}\gamma_{2,1}$ is $\gamma_{1,2;2^3}\gamma_{2,1}^+ + \gamma_{1,2;2^3}\gamma_{2,1}^-$. But H^9 is of rank four, so we must use relations among the six terms in these restrictions. Accounting for transfer maps as in Proposition 5.5 points to no consistently good choices.

When $n > 2$, the behavior of restriction to V_n^+ becomes regular, and more parallel to the symmetric group setting. The invariants to which the cohomology of alternating groups restrict are again the Dickson algebras $\mathbb{F}_2[x_1, \dots, x_n]^{GL_n(\mathbb{F}_2)}$, which are polynomial on generators $d_{k,\ell}$ in dimensions $2^k(2^\ell - 1)$ where $k + \ell = n$. Because the index of the normalizer of V_n^+ in \mathcal{A}_{2^n} is twice that of its image V_n in \mathcal{S}_{2^n} , there are twice as many conjugates by using elements in \mathcal{S}_{2^n} , so V_n^- will not be conjugate to V_n^+ . But because V_n^+ and V_n^- are both conjugate to V_n when included in \mathcal{S}_{2^n} we have the following.

Proposition 5.7. *Restriction maps to V_n^\pm and V_n satisfy the following commutative diagram*

$$\begin{array}{ccc}
H^*(BS_{2^n}) & \xrightarrow{\text{res}} & H^*(BA_{2^n}) \\
\downarrow \text{res} & & \downarrow \text{res} \oplus \text{res} \\
H^*(BV_n) & \longrightarrow & H^*(BV_n^+) \oplus H^*(BV_n^-),
\end{array}$$

where the bottom arrow is the diagonal map.

In the symmetric group setting restriction to products of V_n , including V_1 , detect cohomology. Because $(V_1)^n$ is not a subgroup of \mathcal{A}_n we restrict to the analogous subgroup for alternating groups.

Definition 5.8.

- Let $\mathcal{A}_I = \mathcal{A}_{2^{i_1}} \times \cdots \times \mathcal{A}_{2^{i_q}} \subset \mathcal{A}_{2^m}$, where $|I| = \sum 2^{i_k}$ is equal to m .
- Let $AV_{1,m}$ be the subgroup of \mathcal{A}_{2^m} obtained by intersecting with $(V_1)^m$ in \mathcal{S}_{2^m} .

The full set of maximal elementary abelian subgroups of alternating groups is described in [16]. In Section 7, we show that restriction to only these subgroups along with V_n when appropriate is injective on cohomology. To apply that result requires calculations such as the following.

Theorem 5.9. *The restriction to $AV_{1,m}$ of a Fox-Neuwirth cocycle whose constituent cochains each have two consecutive non-zero terms is zero.*

Proof. We first model the map on classifying spaces induced by inclusion of $AV_{1,m}$ as a subgroup of \mathcal{A}_{2^m} . Note that $AV_{1,m}$ sits in \mathcal{A}_{2^m} as even products of the two-cycles $\tau_{2i} = (\{2i-1\}\{2i\})$. We model $EAV_{1,2m}$ as $(S^\infty)^m$ by having each σ act by -1 on each factor corresponding to a two-cycle τ_{2i} which occurs in σ . This model then includes into our model $\text{Ind}_{2^m}(\mathbb{R}^\infty)$ for $E\mathcal{A}_{2^m}$ by sending

$$(v_1, \dots, v_m) \mapsto (x_1 - \varepsilon v_1, x_1 + \varepsilon v_1, x_2 - \varepsilon v_2, \dots, x_m + \varepsilon v_m).$$

Here we pick some points $x_i \in \mathbb{R}^\infty$ which are linearly independent and do not share their first coordinates and choose ε so that collections of points on spheres of radius ε about x_i also have these two properties.

The composite

$$(S^\infty)^m / AV_{1,2m} \rightarrow \text{Ind}_{2^m}(\mathbb{R}^\infty) / AV_{1,2m} \rightarrow \text{Ind}_{2^m}(\mathbb{R}^\infty) / \mathcal{A}_{2^m}$$

models the map on classifying spaces induced by inclusion of $AV_{1,2m}$ as a subgroup of \mathcal{A}_{2^m} . In this model, the restriction to $C^*(BAV_{1,2m})$ of a cochain on $B\mathcal{A}_{2^m}$ evaluates a chain by composing with this composite, whose image consists of points which all respect sequences of the form $\Gamma = [a_1, 0, a_2, 0, \dots, 0, a_m]$. So a Fox-Neuwirth cochain with two consecutive non-zero terms will restrict to zero. \square

6. GENERATORS, WITH RESPECT TO BOTH PRODUCTS

6.1. Hopf ring generators for \mathcal{B}_+ and \mathcal{B}_- . For \mathcal{A}_{2^n} with $n > 2$, consider the Hopf ring generator $\gamma_{\ell,2^k}$, with $\ell + k = n$ and $\ell > 1$, which is in $H^{2^n - 2^k}(BS_{2^n})$. By Theorem 4.9 of [8] it is represented by the Fox-Neuwirth cochain

$$\alpha_{\ell,2^k} = [1, 1, \dots, 1, 0, 1, 1, \dots, 1, 0, \dots, 0, 1, \dots, 1, 1],$$

which restricts to the sum $\alpha_{\ell,2^k}^+ + \alpha_{\ell,2^k}^-$ in $FNA_{2^n}^*$. The $\alpha_{\ell,2^k}^\pm$ have non-zero boundary, but we will complete each to a cocycle, and extend these to a family of generators $\gamma_{\ell,m}^+ \in H^{m(2^\ell - 1)}(B\mathcal{A}_{m \cdot 2^\ell})$.

Definition 6.1. Let $\alpha_{\ell,m}^\pm \in FNA_{m \cdot 2^\ell}^{m(2^\ell - 1)}$ be the positive (resp. negative) Fox-Neuwirth cochain with m blocks, each a sequence of $2^\ell - 1$ ones, separated by zeros.

Let $\beta_{\ell,m}(i,j)^\circ$ be the sum of positive and negative Fox-Neuwirth cochains each with $m + 1$ blocks so that:

- the i th block is a singleton two;
- the j th block is a sequence of $2^\ell - 3$ ones;

- all other blocks are sequences of $2^\ell - 1$ ones.

Let $\beta_{\ell,m}^o$ denote the sum $\sum \beta_{\ell,m}(i,j)^o$ over all i,j with $1 \leq i < j \leq m+1$.

Let $\gamma_{\ell,m}^+ = \alpha_{\ell,m}^+ + \beta_{\ell,m}^o$, and let $\gamma_{\ell,m}^- = \alpha_{\ell,m}^- + \beta_{\ell,m}^o$.

Proposition 6.2. *The $\gamma_{\ell,m}^\pm$ are cocycles which represent distinct nonzero classes in $H^{m(2^\ell-1)}(BA_{m \cdot 2^\ell})$. When $m = 2^k$, these transfer to $\gamma_{\ell,2^k}$ in the cohomology of symmetric groups.*

Proof. Applying Lemma 4.12, we have that

$$\delta(\alpha_{\ell,m}^+) = \sum_{\{i \neq j \cdot 2^\ell\}} \alpha_{\ell,m}^o \langle i \rangle.$$

Lemma 4.12 also implies that

$$\delta(\beta_{\ell,m}(j, j+1)^o) = \sum_{i=(j-1) \cdot 2^\ell + 1}^{j \cdot 2^\ell - 1} \alpha_{\ell,m}^o \langle i \rangle + \beta_{\ell,m}(j, j+1)^o \langle j \cdot 2^\ell \rangle,$$

as the only terms with non-zero coefficients consist of shuffling the singleton two into the small block of consecutive ones, and concatenating the small block of ones with its neighboring larger block. When the two distinguished blocks are separated, the coboundaries that arise consist of similar collections of non-zero terms, and these cancel in pairs as the distinguished blocks vary, producing telescoping sums so that

$$\delta \left(\sum_{p=j+2}^{m+1} \beta_{\ell,m}(j, p)^o \right) = \beta_{\ell,m}(j, j+1)^o \langle j \cdot 2^\ell \rangle.$$

Summing across all $\delta(\beta_{\ell,m}(j, p)^o)$, we obtain precisely $\delta(\alpha_{\ell,m}^o)$, and so $\gamma_{\ell,m}^\pm$ is a cocycle.

By Theorem 4.14, $\gamma_{\ell,2^k}^\pm$ each transfer to $\gamma_{\ell,2^k}$ in the cohomology of symmetric groups, since the $\beta_{\ell,2^k}(i, j)^o$ vanish under transfer. The cohomology classes they represent must be distinct, since their sum is the restriction of $\gamma_{\ell,2^k}$. \square

By abuse, we will also denote the cohomology classes represented by the $\gamma_{\ell,m}^\pm$ using the same notation. Let $\gamma_{1,2^n}$ denote the restriction of $\gamma_{1,2^n}$.

Proposition 6.3. *For $n \geq 3$, cup monomials of $\gamma_{\ell,2^k}^+$ and $\gamma_{1,2^n;2^n}$ map injectively under the restriction to V_n^+ and, except for powers of $\gamma_{1,2^n;2^n}$ alone, map to zero in V_n^- .*

Proof. Consider the restriction maps in Proposition 5.7. As shown in Section 7 of [8], the image of $\gamma_{\ell,2^k}$ under the vertical restriction map is the Dickson generator $d_{2^n-2^k}$. Because V_n^+ and V_n^- are both conjugate in \mathcal{S}_{2^n} , the restriction to the lower right corner is the corresponding Dickson generator on each factor. Following the diagram in the other direction, this says that the restriction of $\gamma_{\ell,2^k}^+ + \gamma_{\ell,2^k}^-$ for $\ell \geq 2$ must map to this direct sum of Dickson generators. Since involution switches both $\gamma_{\ell,2^k}^+$ and $\gamma_{\ell,2^k}^-$ as well as V_n^+ and V_n^- , and there are no other non-zero classes in this degree, $\gamma_{\ell,2^k}^+$ must map to the Dickson generator in V_n^+ and zero in V_n^- , and vice versa for $\gamma_{\ell,2^k}^-$.

Proposition 5.7 also implies that $\gamma_{1,2^n;2^n}$ restricts to the lowest-degree Dickson generator on both V_n^+ and V_n^- . Thus polynomials in $\{\gamma_{\ell,2^k}^+, \gamma_{1,2^n;2^n}\}$ restrict to polynomials in the corresponding Dickson classes on V_n^+ , and thus form a polynomial ring which maps injectively. \square

We now construct \mathcal{G}_a , which by Theorem 3.8 accounts of the annihilator ideal “half” of the Gysin sequence, through a filtration of the the cohomology of symmetric groups.

Definition 6.4. Define the \odot -partition of a Hopf ring monomial in $H^*(BS_{2^n})$ to be the partition of n by the widths (that is, component numbers divided by two) of the constituent cup monomials.

While \odot -partitions give a direct sum decomposition of the cohomology of symmetric groups, we instead consider the filtration given by partition refinement, which is preserved by multiplication because of Hopf ring distributivity.

For symmetric groups, the subgroups V_n perfectly detect decomposibility with respect to transfer product. We have seen that the $\gamma_{\ell,2^k}$ restrict injectively to generators of the Dickson algebra, and now conversely we have the following.

Theorem 6.5. *Transfer product decomposables in $H^*(BS_{2^n})$ restrict to zero in the cohomology of V_n .*

Proof. This follows immediately from Theorem 7.8 of [8], which implies through the definition of scale- n quotient that decomposables with respect to \odot -product, which have smaller scale, restrict to zero in V_n .

More directly, the image in homology of V_n is exactly Dyer-Lashof generators in $H_*(BS_{2^n})$. Using work of Bruner-May-McClure-Steinberger, namely Theorem 1.5 of [3], we show in Theorem 4.13 of [8] that these Dyer-Lashof generators are primitive with respect to the coproduct dual to the transfer product. As \odot -decomposables evaluate to zero on the image of homology of V_n , \odot -decomposables restrict to zero in the cohomology of V_n . \square

Recall our main strategy, outlined in Section 3.5, to produce an additive basis through the Gysin sequence.

Theorem 6.6. *\mathcal{B}_+ and \mathcal{B}_- are contained in the almost-Hopf ring generated by all $\gamma_{\ell,2^k}^+$ and $\gamma_{1,2^n;2^n}$, along with 1^-*

Proof. We compute how chosen transfer and cup products of these generators map under the transfer map in the Gysin sequence in order to see that their images generate \mathcal{G}_a . Since transfer maps do not preserve cup products, we argue by detection in V_n^\pm .

By Proposition 6.3 a cup-monomial m^+ in $\{\gamma_{\ell,2^k}^+, \gamma_{1,2^n;2^n}\}$ will map to the corresponding Dickson monomial in the cohomology of V_n^+ , and by conjugation the corresponding monomial m^- in $\{\gamma_{\ell,2^k}^-, \gamma_{1,2^n;2^n}\}$ will map to the same monomial in V_n^- . By Proposition 5.7 the image of m^+ under transfer, which restricts to $m^+ + m^-$, must map to that same Dickson monomial in the cohomology of V_n . Since cup-monomials in the $\gamma_{\ell,2^k}$ restrict isomorphically to the Dickson invariants in the cohomology of V_n , and all \odot -decomposables map to zero by Theorem 6.5, the transfer of m^+ must equal the monomial $m \in H^*(BS_{2^n})$ obtained by removing decorations, modulo \odot -decomposables.

Inductively applying Proposition 3.4, a Hopf ring monomial $h = m_1 \odot m_2 \odot \cdots \odot m_i \in \mathcal{G}_a$ will be the image under transfer of the alternating group monomial $h^+ = m_1^+ \odot m_2^+ \odot \cdots \odot m_i^+$, modulo terms with finer \odot -partitions. The induction reduces to BS_4 and BA_4 where the elements of \mathcal{G}_a are the transfer image of products of our $\gamma_{2,1}^\pm$ and $\gamma_{1,2;2}$ in \mathcal{A}_4 as explicitly shown in Proposition 5.5. The pair h^+ and $h^- = 1^- \odot h^+$, which are Hopf ring monomials in the stated generators, thus inductively account for $h \in \mathcal{G}_a$ in the Gysin sequence, which means they generate \mathcal{B}_+ and \mathcal{B}_- . \square

While we argue by filtration here, it follows from Theorem 8.1 that cup-monomials in the $\{\gamma_{\ell,2^k}^+, \gamma_{1,2^n;2^n}\}$ for $\ell > 2$ transfer to exactly the corresponding monomials in \mathcal{G}_a . Proposition 5.5 shows this is not the case for $\ell = 2$.

6.2. Hopf ring generators for \mathcal{B}_o . We next account for subset of \mathcal{G}_q generated by the $\gamma_{1,m}$, before moving on to $\mathcal{G}_q \setminus \mathcal{G}_a$ in general. Applying Proposition 7.2 of [8], on a fixed component \mathcal{S}_{2^n} the subspace of the cohomology of a symmetric group generated by the $\gamma_{1,m}$ as Hopf ring is polynomial, generated by the collection of $\gamma_{1,k} \odot 1_{m-k}$. Indeed, this subset of $H^*(BS_{2m})$ restricts isomorphically to the ring of symmetric polynomials in the cohomology of $H^*(BS_2 \times \cdots \times BS_2)$, a fact we will adapt for alternating groups.

Definition 6.7. For $k \geq 2$ let $\gamma_{1,k;m} \in H^k(\mathcal{A}_{2m})$ be the image of $\gamma_{1,k} \odot 1_{m-k}$ under restriction.

Since restriction is a ring map, these classes generate the image of the scale-one subset \mathcal{G}_q on \mathcal{A}_m under cup product.

Theorem 6.8. \mathcal{B}_o is contained in the almost-Hopf ring generated by the classes $\gamma_{\ell,2^k}^+$, $\gamma_{1,k;m}$, and 1_m .

Proof. An arbitrary element of \mathcal{G}_q is of the form $m_1 \odot m_2$ where m_1 has scale one and m_2 has scale greater than or equal to two. We know that m_1 is a cup-monomial in $\gamma_{1,k;d}$ for d the width of m_1 (this includes 1_d as a trivial cup-monomial). By Theorem 6.6, m_2 is the image under transfer of a sum of Hopf ring monomials in $\gamma_{\ell,2^k}^\pm$ and $\gamma_{1,2^n;2^n}$. Applying Proposition 3.6, we have that the transfer product of m_1 with that sum is the image under restriction of $m_1 \odot m_2$. \square

Along with Theorem 6.6 this implies the following.

Corollary 6.9. $\gamma_{\ell,2^k}^+$ and $\gamma_{1,k;m}$ along with the 1_m and 1^- are almost Hopf ring generators for $H^*(BA_\bullet)$.

In the cohomology of symmetric groups, we have Hopf ring generators $\gamma_{1,k} \in H^k(S_{2k})$ and then all of our scale-one cup ring generators on different components are the transfer products of these with unit classes. By Proposition 3.14, classes pulled back from the cohomology of symmetric groups such as the $\gamma_{1,k;m}$ and the unit classes annihilate each other under transfer product, so in fact these $\gamma_{1,k;m}$ are \odot -indecomposable. Having such ‘‘extra’’ Hopf ring generators is a minor nuisance.

7. DETECTION BY SUBGROUPS

As often the case in group cohomology we manage relations by restricting to cohomology of subgroups, showing that restriction to those defined in Section 5 is injective.

Theorem 7.1. The mod-two cohomology of \mathcal{A}_{2^m} for m not a power of two is detected by the subgroups \mathcal{A}_I , over all I with $|I| = m$, and $AV_{1,m}$. The cohomology of \mathcal{A}_{2^n} is detected by \mathcal{A}_I , $AV_{1,2^n-1}$, and V_n^\pm .

Starting with the embedding of the cohomology of \mathcal{A}_4 in that of V_2 in Theorem 5.2, we inductively deduce the following.

Corollary 7.2. The mod-two cohomology of alternating groups is detected on elementary abelian subgroups. Thus by Quillen’s theorem, there are no nilpotent elements in these cohomology rings.

This theorem and corollary are analogues of similar statements for symmetric groups originally due to Madsen and Milgram [13]. The analogue of Theorem 7.1 for symmetric groups only involves the coproduct and restriction to V_n , as restriction to $(V_1)^n$ occurs inductively.

Proof of Theorem 7.1. The proof will be through theorems we establish below. We proceed through analysis of the restriction of our additive basis to each family of subgroups. The ‘‘matrix’’ defined through these restrictions is the following.

	V_n^\pm	$\coprod \mathcal{A}_I$	$AV_{1,n}$
Cup monomials in \mathcal{B}_\pm	Injective by Theorem 6.3	Follows from (9) and (10) of Theorem 8.1	Mostly 0 by Theorem 7.8
\odot -decomposables in \mathcal{B}_\pm and \mathcal{B}_o	0 by Theorem 7.3	Injective by Theorem 7.5	–
Scale one in \mathcal{B}_o	0 by Proposition 7.4	\otimes Scale One by Proposition 7.6	Injective by Theorem 7.7.

We will show in Theorem 7.3 and Propositions 7.4 and 7.6 that the matrix representing all of these restrictions is effectively block upper-triangular, after quotienting the cohomology of \mathcal{A}_I by the span of the scale-one subset of \mathcal{B}_o . We show that the homomorphisms corresponding to the diagonal blocks are injective in Theorems 6.3, 7.5 and 7.7, and moreover in Theorem 7.5 that in fact coproduct continues to be injective after quotienting by scale-one subset. \square

Theorem 7.3. *The restriction to V_n^\pm of transfer product decomposables are zero.*

Proof. Transfer products involving cup-monomials in the $\gamma_{1,k;m}$ will be pulled back from the cohomology of symmetric groups, from which the result follows from Theorem 6.5. All of our other almost-Hopf ring generators are in the cohomology of alternating groups indexed by powers of two, so a non-trivial transfer product in our basis \mathcal{B}_\pm , \mathcal{B}_o factors through $\mathcal{A}_{2^{n-1}} \times \mathcal{A}_{2^{n-1}}$.

We apply Proposition 2.4 for $G = \mathcal{A}_{2^n}$, $H = \mathcal{A}_{2^{n-1}} \times \mathcal{A}_{2^{n-1}}$ and $K = V_n^+$. In this case, $B\iota_K^* \circ B\iota_H^!$ is the restriction to V_n of a transfer product. Because the composite of natural and restriction maps commute for pull-backs, this composite coincides with $g^! \circ f^*$ in the notation of Proposition 2.4, which we show is zero by producing an involution on the pull-back.

Recall in general that any $\sigma \in G$ with $\sigma H = H\sigma$ defines a permutation of double-cosets by $HgK \mapsto H(\sigma g)K$. Here we choose σ to be the product of 2-cycles $(1\ 2)(\{2^{n-1} + 1\}\{2^{n-1} + 2\})$, which obviously normalizes $H = \mathcal{A}_{2^{n-1}} \times \mathcal{A}_{2^{n-1}}$. We claim that σ permutes all double-cosets and thus components of the pull-back non-trivially. From the definition, a fixed coset indexed by g would coincide with a non-trivial intersection between the right-coset $H\sigma$ and the conjugate gKg^{-1} . All elements of $K = V_n$ and thus its conjugates are involutions of the form $(i_1 i_2)(i_3 i_4) \cdots (i_{2^{n-1}-1} i_{2^n})$, which we call a full product of transpositions. Because σ preserves the partition into the first 2^{n-1} and last 2^{n-1} elements, and by definition $H = \mathcal{A}_{2^{n-1}} \times \mathcal{A}_{2^{n-1}}$, $H\sigma$ preserves the partition as well. So in order for an element of $(\tau_1 \times \tau_2) \cdot \sigma$ to be a full product of transpositions, we would have that $\tau_1 \cdot (1\ 2)$ and $\tau_2 \cdot (\{2^{n-1} + 1\}\{2^{n-1} + 2\})$ would themselves be full products of transpositions, though of course half as long. But such permutations are even, so τ_1 and τ_2 could not be even as well.

Because the image of f^* is invariant under the involution defined by σ with no fixed components, and $g^!$ will send each in a pair of cohomology classes (or even cochains) to the same image, the composite $g^! \circ f^*$ is zero on mod-two cohomology, proving the result for V_n^+ . The result for V_n^- follows by applying the involution, or from similar analysis. \square

While this theorem is the direct analogue for alternating groups of Theorem 6.5 for symmetric groups, and V_n is a subgroup of both, we could not find a unified line of argument. Gysin sequence calculations and use of the involution have not established the alternating group statement as a corollary of the symmetric group case. The original proof for symmetric groups uses facts about homology - that is, the Dyer-Lashof algebra - which are not known for alternating groups. And the proof we give here for alternating groups does not translate to the symmetric group setting, as the normalizer of $H = \mathcal{S}_{2^{n-1}} \times \mathcal{S}_{2^{n-1}}$ is contained in the identity double-coset of $H \backslash \mathcal{S}_{2^n} / V_n$.

In contrast to this, our knowledge of restrictions for symmetric groups does immediately lead to the following.

Proposition 7.4. *Scale one generators $\gamma_{1,k;2^n}$ with $k < 2^n$ map to zero under restriction to V_n^\pm .*

Proof. By definition, these generators are the restriction of a \odot -decomposable class, namely $\gamma_{1,k} \odot 1_{2^n-k}$. They thus share their restriction to V_n , which is zero by Theorem 6.5. \square

We move on to consider the restriction to arbitrary products of alternating groups. By definition, the coproduct Δ is the map induced by the embedding of $\mathcal{A}_n \times \mathcal{A}_m$ in \mathcal{A}_{n+m} . So we set Δ_I to be the restriction map to \mathcal{A}_I . Define the scale one subspace of \mathcal{A}_I to be the tensor product of scale one subspaces of constituent \mathcal{A}_{2^i} (that is, all tensor factors are scale one).

Theorem 7.5. *The transfer product decomposables in \mathcal{B}_\pm and \mathcal{B}_o map injectively under $\bigoplus \Delta_I$, and continue to do so after quotienting by the scale one subspaces of the cohomology of the \mathcal{A}_I .*

Proof. Our basis elements which are \odot -decomposable are of the form $h = m_1^+ \odot \cdots \odot m_p^+ \odot m_\omega$, where the m_i are cup monomials of our generators and m_ω is scale one or could be 1^- . Let w_i be the width of m_i (including $i = \omega$) and let $I_h = \{w_i\}$.

By repeated application of Theorem 3.21, $\Delta_{I_h}(h)$ will, in the basis given by tensors products of our standard bases, contain exactly one term which is a tensor product of cup-monomials, namely $m_1^+ \otimes \cdots \otimes m_p^+ \otimes m_\omega$. Thus, all h with I_h equal to a given I map injectively under Δ_I , and that continues to be the case after quotienting by the scale one subspace of \mathcal{A}_I since m_1^+ is not scale one. Taking direct sum over possible I_h gives the result. \square

We finish proving our detection result by addressing scale-one basis elements. The next result is immediate from Proposition 3.4 and the corresponding fact for symmetric groups, since all scale-one elements are pulled back from symmetric groups.

Proposition 7.6. *Scale one basis elements map to tensor product of scale one basis elements under the coproduct restriction to \mathcal{A}_I .*

Unfortunately, there are scale-one classes whose coproduct and restriction to V_n (when applicable) are trivial, including all $\gamma_{1,3;m}$. We detect such classes using the alternating groups versions of subgroups which detect scale-one classes for symmetric groups, as described in Theorem 7.8 of [8].

Theorem 7.7. *Scale one classes define a polynomial subring which restricts injectively to $AV_{1,m}$.*

Proof. Scale-one classes are pulled back from symmetric groups, with $\gamma_{1,k;m} = \text{res}(\gamma_{1,k} \odot 1_{m-k})$. The restriction from \mathcal{S}_{2m} to $AV_{1,m}$ factors through $(V_1)^m$, the subgroup of \mathcal{S}_{2m} generated by the two-cycles $(\{2k-1\}\{2k\})$. There, by Theorem 7.8 of [8] the restriction of $\gamma_{1,k} \odot 1_{m-k}$ in the cohomology of $(V_1)^m$ is σ_k , the k th symmetric polynomial.

So we calculate the restriction from $(V_1)^m$ to $AV_{1,m} \cong (C_2)^{m-1}$, whose cohomology rings are polynomials in m and $m-1$ variables respectively. Choose generators of $AV_{1,m}$ as $(\{2k-1\}\{2k\})(\{2m-1\}\{2m\})$, which we call $\tau_{2k}\tau_{2m}$. Then let x_k be the generator of cohomology of $B(V_1)^m$ corresponding to the two-cycle τ_{2k} and y_k be the generator of the cohomology of $BAV_{1,m}$ corresponding to $\tau_{2k}\tau_{2m}$. The restriction homomorphism is then given by

$$x_i \mapsto y_i \quad i < m; \quad x_m \mapsto \sum_{i=1}^{m-1} y_i.$$

The image of $\gamma_{1,k;m}$ is the image of the k th symmetric function in the x_i under this homomorphism. Because

$$\sigma_i(x_1, \dots, x_m) = \sigma_i(x_1, \dots, x_{m-1}) + \sigma_{i-1}(x_1, \dots, x_{m-1}) \cdot x_m,$$

this homomorphism sends

$$\sigma_i(x_1, \dots, x_m) \mapsto \sigma_i(y_1, \dots, y_{m-1}) + \sigma_{i-1}(y_1, \dots, y_{m-1}) \cdot \sigma_1(y_1, \dots, y_{m-1}).$$

Thus $\gamma_{1,k;m}$ are sent to polynomial ring generators modulo decomposables, so their image a polynomial ring. The subring generated by $\gamma_{1,k;m}$ must itself be polynomial, mapping isomorphically to its image under restriction. \square

This analysis completes our proof of Theorem 7.1.

In order to apply our detection result to verify relations, we need one further calculation of a restriction map to $AV_{1,2^n}$. Restriction calculations which are coproducts will be completed below.

Theorem 7.8. *The restriction of $\gamma_{\ell,m}^\pm$ to $AV_{1,m2^{\ell-1}}$ is zero for $\ell \geq 2$ other than $\ell = 2, m = 1$.*

Proof. We apply Theorem 5.9. For $\ell > 2$, the Fox-Neuwirth cochain representative of $\gamma_{\ell,2^k}$ as given in Definition 6.1 all have at least five 1's in each block. For $\ell = 2, m > 1$ the representative has three consecutive 1's in some block. \square

8. PRESENTATION OF THE COHOMOLOGY OF ALTERNATING GROUPS

Theorem 8.1. $H^*(BA_\bullet)$ is the almost-Hopf ring under cup and transfer products generated by classes

$$\begin{aligned}\gamma_{\ell,m}^+ &\in H^{m(2^\ell-1)}(BA_{m \cdot 2^\ell}) \quad 2 \leq \ell, 1 \leq m, \\ \gamma_{1,k;m} &\in H^k(BA_{2m}) \quad 2 \leq k \leq m, \\ 1_m &\in H^0(BA_{2m}) \quad 1 \leq m, \text{ and } 1^\pm \in H^0(BA'_0),\end{aligned}$$

where the 1_m are units for cup products on their components and 1^+ is the unit for transfer product.

Relations between transfer products are

$$\begin{aligned}(1) \quad & \gamma_{\ell,m}^+ \odot \gamma_{\ell,n}^+ = \binom{m+n}{n} \gamma_{\ell,m+n}^+ \\ (2) \quad & 1^- \odot 1^- = 1^+ \\ (3) \quad & (1^+ + 1^-) \odot \gamma_{1,k;m} = 0 \\ (4) \quad & \prod \gamma_{1,k;m} \odot \prod \gamma_{1,\ell;n} = 0,\end{aligned}$$

where the products of $\gamma_{1,k;m}$ are arbitrary cup products which by convention include the empty product 1_m .

Cup products between classes on different components are zero, and further cup relations are

$$\begin{aligned}(5) \quad & \gamma_{\ell,m}^+ \cdot \gamma_{k,n}^- = 0 \text{ unless } k = \ell = 2, \\ (6) \quad & \gamma_{2,m}^+ \cdot \gamma_{2,m}^- = \begin{cases} (\gamma_{2,m}^+ + \gamma_{2,m}^-)^2 + (\gamma_{2,m-1}^+)^2 \odot (\gamma_{1,2})^3 & \text{if } m \text{ is odd} \\ (\gamma_{2,m-1}^+)^2 \odot (\gamma_{1,2})^3 & \text{if } m \text{ is even,} \end{cases} \\ (7) \quad & \gamma_{\ell,m}^+ \cdot \gamma_{1,k;m2^{\ell-1}} = \begin{cases} (\gamma_{\ell,q}^+ \cdot \gamma_{1,k}) \odot \gamma_{\ell,m-q} & \text{if } k = 2^{\ell-1} \cdot q \\ 0 & \text{if } k \text{ is not a multiple of } 2^{\ell-1}. \end{cases}\end{aligned}$$

While $H^*(BA_\bullet)$ does not form a Hopf ring, for any α and β there is in general the equality

$$(8) \quad \Delta(\alpha \odot \beta) = \Delta\alpha \odot_{\rho^+} \Delta\beta,$$

where \odot_{ρ^+} is transfer product after applying the polarization operator of Theorem 3.21.

Let $\gamma_{\ell,m}^-$ denote $1^- \odot \gamma_{\ell,m}^+$, and by convention set $\gamma_{\ell,0}^\pm = 1^\pm$ and $\gamma_{1,1;m} = 0$. Coproducts of generators are given by

$$(9) \quad \Delta\gamma_{\ell,m}^+ = \sum_{i+j=m} \left(\gamma_{\ell,i}^+ \otimes \gamma_{\ell,j}^+ + \gamma_{\ell,i}^- \otimes \gamma_{\ell,j}^- \right)$$

$$(10) \quad \Delta\gamma_{1,k;m} = \sum \gamma_{1,p;i} \otimes \gamma_{1,q;j},$$

where the last sum is over i, j, p, q with $i + j = m$ and $p + q = k$, where $0 \leq p \leq i$ and $0 \leq q \leq j$.

The appropriate context for understanding this presentation is the closely related presentation given in Theorem 3.1 of the cohomology of symmetric groups, which as a Hopf ring is built from polynomial rings in the $\gamma_{\ell,2^k}$. For alternating groups, we build two copies of these polynomial rings on each \mathcal{A}_{2^k} , though with lowest generator shared. Most products between these copies are zero, except for the $\gamma_{2,m}^\pm$ whose unique behavior is due to the fact that for \mathcal{A}_4 there is only one copy of the transitive maximal elementary abelian 2-subgroup, while for higher \mathcal{A}_{2^k} there are two. Finally, the coproducts and transfer products of these two sets of generators behave according to rules governing charge, with the $\gamma_{1,k;m}$ and 1_m being neutral.

The main results of Section 6 – namely Theorems 6.6 and 6.8 which lead to Corollary 6.9 – show that the classes listed generate the cohomology of $H^*(BA_\bullet)$ under cup and transfer products. We establish the rest of the result presently.

8.1. Coproducts. While listed as Relations (8)-(10), we establish the coproduct calculations first so we can use them in the process of verifying the other relations. Relation (8) is just a restatement of Theorem 3.21. In practice, it means taking only half of the terms of the coproduct of a charged class (there must be one charged class to consider, or else the transfer product will be zero), for example only the $+\otimes+$ terms and not the $-\otimes-$ terms in the coproduct of a positively charged class.

Next using Proposition 3.4 we see that Relation (10) is immediate from the coproduct formula for $\gamma_{1,n}$ in the cohomology of symmetric groups. For Relation (9) on the other hand, the statement for alternating groups implies the statement for symmetric groups, or conversely the statement for symmetric groups implies that for alternating groups modulo neutral classes.

To establish Relation (9) we apply Theorem 4.16 and for convenience we let $\overline{\Delta} = \bigoplus_{i,j>0} \Delta_{i,j}$. For the $m = 1$ case, we consider the cochain level representative for $\gamma_{\ell,1}^\pm$, given in Proposition 6.2,

$$\gamma_{\ell,1}^\pm = [1, 1, \dots, 1]^+ + [2, 0, 1, 1, \dots, 1]^o,$$

The first term has trivial coproduct by Theorem 4.16, while the second decomposes as $[2]^o \otimes [1, 1, \dots, 1]^o$, which is the coboundary of $[1]^+ \otimes [1, 1, \dots, 1]^o$.

To establish the general case, we require two new types of FN cochains. Let $\sigma_{\ell,m}(p; r)^o$ and $\tau_{\ell,m}(i, j)^o$ each be the sum of positive and negative cochains so that

- $\sigma_{\ell,m}(p; r)^\pm$ have $(m + 1)$ blocks, the p th of which is a sequence of r ones, and
- $\tau_{\ell,m}(p, q; r, s)^\pm$ have $(m + 2)$ blocks, the p th of which is a sequence of r ones, the q th of which is a sequence of s ones,

and all other blocks in each consisting of sequences of $2^\ell - 1$ ones. Thus, for example, $\sigma_{2,3}(1; 2)^o = [1, 1, 0, 1, 1, 1, 0, 1, 1, 1]^o$ and $\tau_{3,3}(1, 3; 2, 1) = [1, 1, 0, 1, 1, 1, 1, 1, 1, 0, 1]^o$.

We apply Theorem 4.16. As required,

$$\overline{\Delta}(\alpha_{\ell,m}^+) = \sum_{i+j=m} \left(\alpha_{\ell,i}^+ \otimes \alpha_{\ell,j}^+ + \alpha_{\ell,i}^- \otimes \alpha_{\ell,j}^- \right).$$

However $\overline{\Delta}(\beta_{\ell,m}^o)$ contains several additional terms, as follows

$$\begin{aligned} \overline{\Delta}(\beta_{\ell,m}(p, q)^o) &= \sum_{\substack{i+j=m \\ 1 \leq i < p}} \alpha_{\ell,i}^o \otimes \beta_{\ell,j}(p-i, q-i)^o + \sum_{\substack{i+j=m \\ q \leq i \leq m}} \beta_{\ell,i}(p, q)^o \otimes \alpha_{\ell,j}^o \\ &\quad + \sum_{\substack{i+j=m \\ p \leq i < q}} \sigma_{\ell,i}(p; 1) \langle (p-1)2^\ell + 1 \rangle^o \otimes \sigma_{\ell,j}(q-i; 2^\ell - 3)^o. \end{aligned}$$

For example,

$$\begin{aligned} \overline{\Delta}(\beta_{2,4}(2, 4)^o) &= [1, 1, 1]^o \otimes [2, 0, 1, 1, 1, 0, 1]^o + [1, 1, 1, 0, 2]^o \otimes [1, 1, 1, 0, 1]^o + [1, 1, 1, 0, 2, 0, 1, 1, 1]^o \otimes [1]^o \\ &= \alpha_{2,1}^o \otimes \beta_{2,3}(1, 3)^o + \sigma_{2,2}(2; 1) \langle 2 \rangle^o \otimes \sigma_{2,2}(2; 1)^o + \sigma_{2,3}(2; 1) \langle 2 \rangle^o \otimes \sigma_{2,1}(1; 1)^o. \end{aligned}$$

To obtain the desired relation on cohomology, we produce cochains whose coboundary are precisely these σ terms.

We first compute $\delta(\tau_{\ell,m+1}(p, q; 1, 2^\ell - 5)^o)$ for $p < q$. Let $\kappa_{a,b}$ be the Kronecker delta function and $\overline{\kappa_{a,b}} = 1 - \kappa_{a,b}$. Applying Lemmas 4.12 and 4.13, and using the fact that when $\ell > 2$, $|\text{Sh}_+(2^\ell - 4, 2)|$ is odd and $|\text{Sh}_-(2^\ell - 4, 2)|$ is even, we have that the coboundary of $\tau_{\ell,m+1}(p, q; 1, 2^\ell - 5)^o$ is equal to

$$\overline{\kappa_{p,1}} \tau_{\ell,m}(p-1, q-1; 2^\ell + 1, 2^\ell - 5)^o + \sigma_{\ell,m}(p; 2^\ell - 3)^o + \overline{\kappa_{q,m}} \tau_{\ell,m}(p, q; 1, 2^{\ell+1} - 5)^o,$$

if $q = p + 1$. Otherwise if $q < p + 1$, it is equal to

$$\begin{aligned} & \overline{\kappa_{p,1}} \tau_{\ell,m}(p-1, q-1; 2^\ell + 1, 2^\ell - 5)^o + \tau_{\ell,m}(p, q-1; 2^\ell + 1, 2^\ell - 5)^o \\ & \quad + \tau_{\ell,m}(p, q-1; 1, 2^{\ell+1} - 5)^o + \overline{\kappa_{q,m}} \tau_{\ell,m-1}(p, q; 1, 2^{\ell+1} - 5)^o. \end{aligned}$$

Summing over p and q the resulting τ terms in the boundary telescope, and we have

$$\delta \left(\sum_{1 \leq p < q \leq m+1} \tau_{\ell,m+1}(p, q; 1, 2^\ell - 5)^o \right) = \sum_{p=1}^m \sigma_{\ell,m}(p; 2^\ell - 3)^o.$$

Using the same techniques,

$$\begin{aligned} \delta \left(\sigma_{\ell,m}(p; 1) \langle (p-1)2^\ell + 1 \rangle^o \right) &= \overline{\kappa_{p,1}} \sum_{k=1}^{2^\ell+1} \sigma_{\ell,m-1}(p-1; 2^\ell + 1) \langle (p-2)2^\ell + k \rangle \\ & \quad + \overline{\kappa_{p,m}} \sum_{k=1}^{2^\ell+1} \sigma_{\ell,m-1}(p; 2^\ell + 1) \langle (p-1)2^\ell + k \rangle. \end{aligned}$$

These terms again telescope as we sum over the index p , so $\delta(\sum_{p=1}^m \sigma_{\ell,m}(p; 1) \langle (p-1)2^\ell + 1 \rangle^o) = 0$.

Finally, fixing indices i and j with $i + j = m$ and summing over pairs p and q which produce σ terms in the the coproduct computation for $\beta_{\ell,m}^o$, we have by the Leibniz rule that

$$\begin{aligned} & \delta \left(\sum_{1 \leq p \leq i} \sigma_{\ell,i}(p; 1) \langle (p-1)2^\ell + 1 \rangle^o \otimes \sum_{i+1 \leq s < q \leq j+1} \tau_{\ell,j+1}(s-i, q-i; 1, 2^\ell - 5)^o \right) \\ & \quad = \sum_{1 \leq p \leq i} \sigma_{\ell,i}(p; 1) \langle (p-1)2^\ell + 1 \rangle^o \otimes \sum_{1 \leq q \leq j} \sigma_{\ell,j}(q-i; 2^\ell - 3)^o. \end{aligned}$$

We conclude that

$$\overline{\Delta}(\beta_{\ell,m}^o) = \sum_{i+j=m} \alpha_{\ell,i}^o \otimes \beta_{\ell,j}^o + \beta_{\ell,i}^o \otimes \alpha_{\ell,j}^o + \delta\omega,$$

for the cochain ω determined above, and thus that

$$\begin{aligned} \overline{\Delta}\gamma_{\ell,m}^+ &= \sum_{i+j=m} \left(\alpha_{\ell,i}^+ \otimes \alpha_{\ell,j}^+ + \alpha_{\ell,i}^- \otimes \alpha_{\ell,j}^- \right) + \left(\alpha_{\ell,i}^o \otimes \beta_{\ell,j}^o + \beta_{\ell,i}^o \otimes \alpha_{\ell,j}^o \right) + \delta\omega \\ &= \sum_{i+j=m} \left((\alpha_{\ell,i}^+ + \beta_{\ell,i}^o) \otimes (\alpha_{\ell,j}^+ + \beta_{\ell,j}^o) + (\alpha_{\ell,i}^- + \beta_{\ell,i}^o) \otimes (\alpha_{\ell,j}^- + \beta_{\ell,j}^o) \right) + \delta\omega \\ &= \sum_{i+j=m} \left(\gamma_{\ell,i}^+ \otimes \gamma_{\ell,j}^+ + \gamma_{\ell,i}^- \otimes \gamma_{\ell,j}^- \right) + \delta\omega. \end{aligned}$$

8.2. Transfer product relations. Most transfer product relations are immediate from results in Sections 2 and 3. Recall that by Definition 3.18 the transfer product with 1^- is the conjugation on the cohomology of $B\mathcal{A}_n$ as a two-sheeted cover over $B\mathcal{S}_n$. Relation (2) is just a rephrasing of the fact that this conjugation is an involution. Relation (3) expresses the fact that $\gamma_{1,k;m}$ is fixed under conjugation, as it is pulled back from the cohomology of symmetric groups. Relation (4) follows from Proposition 3.14.

What requires further argument is Relation (1), which we recall is that $\gamma_{\ell,m}^+ \odot \gamma_{\ell,n}^+ = \binom{m+n}{n} \gamma_{\ell,m+n}^+$. We explicitly compute with Fox-Neuwirth cochains from Definition 6.1, with

$$\begin{aligned} \gamma_{\ell,m}^+ \odot \gamma_{\ell,n}^+ &= \left(\alpha_{\ell,m}^+ + \beta_{\ell,m}^o \right) \odot \left(\alpha_{\ell,n}^+ + \beta_{\ell,n}^o \right) \\ &= \alpha_{\ell,m}^+ \odot \alpha_{\ell,n}^+ + \alpha_{\ell,m}^+ \odot \beta_{\ell,n}^o + \beta_{\ell,m}^o \odot \alpha_{\ell,n}^+ + \beta_{\ell,m}^o \odot \beta_{\ell,n}^o \end{aligned}$$

Applying Theorem 4.17, the first term is given by the sum of the shuffles of the zero-blocks of $\alpha_{\ell,m}^+$ and $\alpha_{\ell,n}^+$, each of which results in a copy of $\alpha_{\ell,m+n}^+$, so $\alpha_{\ell,m}^+ \odot \alpha_{\ell,n}^+ = \binom{m+n}{n} \alpha_{\ell,m+n}^+$.

The second and third terms consist of cochains obtained by shuffling an additional m blocks of $2^\ell - 1$ ones into the blocks of cochains of the form $\beta_{\ell,n}(i, j)^o$. These shuffles preserve charge, as the associated permutation at the level of labeled configurations is even, so the transfer product produces cochains of the form $\beta_{\ell,m+n}(i', j')^o$. To compute coefficients, consider the entire product $\alpha_{\ell,m}^+ \odot \beta_{\ell,n}^o$ at once. For any $i' < j'$ and $(m, n-1)$ -shuffle there is a choice of $i < j$ so that the resulting term in the product is $\beta_{\ell,m+n}(i', j')$: simply “unshuffle” to determine what i and j must be. Thus, $\alpha_{\ell,m}^+ \odot \beta_{\ell,n}^o = \binom{m+(n-1)}{n-1} \beta_{\ell,m+n}^o$, and similarly for the third term.

The final term $\beta_{\ell,m}^o \odot \beta_{\ell,n}^o$ is the transfer product of two neutral cochains, and thus is zero by Corollary 4.18.

Combining these, we have

$$\gamma_{\ell,m}^+ \odot \gamma_{\ell,n}^+ = \binom{m+n}{n} \alpha_{\ell,m+n}^+ + \binom{m+(n-1)}{n-1} \beta_{\ell,m+n}^o + \binom{(m-1)+n}{m-1} \beta_{\ell,m+n}^o = \binom{m+n}{n} \gamma_{\ell,m+n}^+.$$

8.3. Cup product relations. To establish cup product relations we apply our main detection result, with coproducts are among the detection homomorphisms. We start with Relation (5) and the case of the product $\gamma_{n,1}^+ \cdot \gamma_{k,2^{n-k}}^-$ for $n > 2$. In the proof of Proposition 6.3, we showed that $\gamma_{n,1}^+$ mapped to zero on V_n^- and that $\gamma_{k,2^{n-k}}^-$ mapped to zero on V_n^+ . Thus their product maps to zero on both of these subgroups. By Relation (9), the the coproduct of $\gamma_{n,1}^+$ is trivial, and as $n > 2$ the restriction of both classes to $AV_{1,2^{n-1}}$ is zero by Theorem 7.8. Because $\gamma_{n,1}^+ \gamma_{k,2^{n-k}}^-$ restricts to zero on V_n^\pm , \mathcal{A}_I and $AV_{1,2^{n-1}}$, it is zero by Theorem 7.1.

The argument extends inductively for $\gamma_{\ell,m}^+ \cdot \gamma_{k,n}^-$ more generally, where the restrictions to \mathcal{A}_I are calculated by repeated application of Relation (9). As (\cdot, Δ) form a bialgebra, this will be zero by induction. The rest of the argument applies to see restrictions to $AV_{1,m2^{\ell-1}}$ and, if applicable, $V_{\ell+p}^\pm$ (where $p = \log_2(m)$) are zero in order to apply Theorem 7.1.

The first case, $m = 1$, of Relation (6) is established in Proposition 5.4. We prove the other cases by induction. Let $\gamma_{2,i}^o = \gamma_{2,i}^+ + \gamma_{2,i}^-$. Multiplying the coproduct of $\gamma_{2,m}^+$ with that of $\gamma_{2,m}^-$ and using Relation (6) inductively, when n is odd we get the sum

$$(\gamma_{2,i}^o)^2 \otimes \left(\gamma_{2,m-i}^o{}^2 + \gamma_{2,m-i-1}^+{}^2 \odot \gamma_{1,2^3} \right) + \left(\gamma_{2,m-i}^o{}^2 + \gamma_{2,i-1}^+{}^2 \odot \gamma_{1,2^3} \right) \otimes (\gamma_{2,m-i}^o)^2.$$

And when n is even we get

$$(\gamma_{2,i}^o)^2 \otimes \left(\gamma_{2,m-i-1}^+{}^2 \odot \gamma_{1,2^3} \right) + \left(\gamma_{2,i-1}^+{}^2 \odot \gamma_{1,2^3} \right) \otimes (\gamma_{2,m-i}^o)^2,$$

a key point being that terms $(\gamma_{2,i}^o)^2 \otimes (\gamma_{2,m-i}^o)^2$ cancel when both i and $n-i$ are odd. These agree with the corresponding coproducts of the right hand side of Relation (6).

We next show that the restriction to $AV_{1,2m}$ of both sides of Relation (6) are zero when $m \geq 2$. The vanishing of the restriction of the left-hand side is immediate from Theorem 7.8. For $(\gamma_{2,m-i-1}^+{}^2 \odot \gamma_{1,2^3})$, we first show that in general $(\gamma_{2,n}^+)^2$ has Fox-Neuwirth representative which is the sum of $\alpha_{2,m}(2)$, which

has m blocks of three repeated 2's, with $\sum_{i=1\dots m} [2, 2, 2, 0, \dots, 0, 3, 1, 2, 0, \dots, 0, 2, 2, 2]^o$, where the i th block of the i th term is $[3, 1, 2]$, modulo further potential neutral terms.

That these are cocycles is straightforward. We show in Proposition A.1 that $\gamma_{2,m}^{2^2}$ is represented by $\alpha_{2,m}(2)$. We deduce from the Gysin sequence $\gamma_{2,n}^{+2} + \gamma_{2,n}^{-2}$ is represented by $\alpha_{2,m}(2)^o$. Since $\gamma_{2,n}^{+2}$ and $\gamma_{2,n}^{-2}$ are related by conjugation, they must be of the form $\alpha_{2,m}(2)^\pm$ plus neutral terms. These terms are as given, but their form is immaterial because we are taking the transfer product with $\gamma_{1,2;2^3}$, which so all neutral terms will contribute zero to the transfer product by Corollary 4.18. Now applying Theorems 4.17 we see that the remaining terms are shuffles of $[2, 2, 2]$ blocks and $[3]$ blocks, which will restrict trivially to $AV_{1,2m}$ by Theorem 5.9 because of the consecutive 2's.

Finally, when $2m$ is a power of two which is greater than four we claim that both sides of Relation (6) restrict to zero on V_n^\pm . Each factor in the left hand side is zero on one of these subgroups as in our proof of Relation (5), and the right-hand-side is a non-trivial transfer product so Theorem 7.3 applies.

Relation (7) is akin to a relation in the symmetric groups setting which follows from Hopf ring distributivity, but needs to be addressed on its own here because the $\gamma_{1,k;n}$ are \odot -indecomposable. The proof is similar to that of Relation (6). The coproduct calculation is straightforward, and the restriction of both sides to V_n^\pm is zero by Theorems 6.5 and 7.3. The left-hand side of the relation restricts to zero on $AV_{1,m}$, by Theorem 7.8. For the right-hand-side, we compute a Fox-Neuwirth representative of $(\gamma_{\ell,q}^+ \cdot \gamma_{1,k}) \odot \gamma_{\ell,m-q}$. We show in Proposition A.1 that a representative of $\gamma_{\ell,q}^+ \cdot \gamma_{1,k}$ is $\alpha_{\ell,m}(1.5)$, which has m blocks of the form $[2, 1, 2, \dots, 1, 2]$, each of length $2^\ell - 1$. Thus a representative for $\gamma_{\ell,q}^+ \cdot \gamma_{1,k}$ is

$$\alpha_{\ell,m}(1.5)^+ + \sum_{1 \leq i < j < m+1} B_m(i, j)^o,$$

where $B_m(i, j)$ also has $[2, 1, 2, \dots, 2]$ blocks along with a $[3]$ block and a $[2]$ block. The resulting transfer products will have $[2, 1, 2, \dots, 2]$ or $[1, 1, \dots, 1]$ blocks (or both), of length at least three, and thus restrict trivially to $AV_{1,2m}$ by Theorem 5.9.

8.4. Completeness of relations. In the course of proving Theorems 6.6 and 6.8, we found an additive basis of Hopf ring monomials, namely those of the form $1^\pm \odot m_1 \odot \dots \odot m_p \odot m_\nu$ where each m_i is a monomial in the $\gamma_{\ell,2^k}^+$ and $\gamma_{1,2^{k+\ell}}^o$, or a chosen representative monomial in the $\gamma_{2,1}^\pm$ and $\gamma_{1,2}^o$ (see Remark 5.6), and where m_ν is a monomial in the of $\gamma_{1,k;m}^o$, possibly empty and possibly with $m = 0$. If m_ν has $m > 0$ (including if empty which means equal to 1_m^o) then 1^\pm is chosen to be 1^+ .

Hopf ring distributivity provides all that is needed to reduce to the chosen Hopf monomial basis in the symmetric groups setting, since the ‘‘building block’’ cup product algebras are polynomial. Here we must show that an arbitrary Hopf monomial can be further reduced. By Relation (1), we can focus on cup monomials of width powers of two, except for cup monomials in only the $\gamma_{1,k;m}^o$, of which there can be at most one by Relation (4).

Relation (7) can be applied so that in any monomials with both $\gamma_{1,m;2^{k+\ell}}^o$, with $m < 2^{k+\ell}$ and $\gamma_{\ell,2^k}^+$ can be reduced to transfer products of monomials where only $\gamma_{1,2^p;2^p}$ occur, applying Relation (1) as needed. Relations (5) and (6) can be applied to reduce to monomials with all generators of width greater than two purely positive or negative. Here we are applying Relation (6) only in the m even setting where additional ‘‘mixed’’ terms cannot arise, except ultimately for $m = 1$. By Proposition 5.4, Relation (6) when $m = 1$ suffices to reduce any width-two monomial. We can ‘‘factor out’’ the $1^- \odot$ from monomials in negative generators by Hopf ring distributivity, and then by Relation (2) there will ultimately be either 1^+ or 1^- multiplying monomials in only positive or neutral generators. By Relation (3), we may assume this is 1^+ if there is a cup product monomial in the $\gamma_{1,k;m}$.

9. STEENROD ACTION

The action of the Steenrod algebra on the mod-two cohomology of alternating groups also parallels that of symmetric groups. Because transfers are stable maps, they preserve Steenrod squares, so the external Cartan formula for Steenrod operations yields one for the \odot -product. The Steenrod algebra structure on $H^*(BA_\bullet)$ is thus determined by the action on Hopf ring generators.

The description of Steenrod operations on most of our Hopf ring generators parallels that given in Definition 8.2 and Theorem 8.3 of [8], but we translate that language of outgrowth monomials into more explicit formulae using partitions, which better serve in accounting for irregularities.

Definition 9.1. A level- ℓ bi-partition π of (j, m) is an equality in $\mathbb{N} \oplus \mathbb{N}$ of the form $(j, m) = \sum c_{p,\ell'}(\pi) \vec{x}_{p,\ell'}$, where $\vec{x}_{p,\ell'} = (2^p(2^\ell - 2^{\ell'}) + 2^p - 1, 2^p)$, the $c_{p,\ell'}(\pi)$ are nonnegative integers and where either $p > 0$ and $0 < \ell' \leq \ell$ or $p = 0$ and $0 \leq \ell' \leq \ell$.

In the following, let $\gamma_{0,2m}^+$ be the unit class in the cohomology of \mathcal{A}_{2m} and any $\gamma_{\ell,0}^+$ be 1^+ .

Theorem 9.2. For $\ell \geq 3$, $Sq^j \gamma_{\ell,m}^+$ is the sum over all level- ℓ bi-partitions π of (j, m) of Hopf ring monomials h_π , where h_π is the transfer product over all $\vec{x}_{p,\ell'}$ of $\gamma_{\ell+p,c_{p,\ell'}(\pi)}^+ \gamma_{\ell-\ell',c_{p,\ell'}(\pi) \cdot 2^{p+\ell'}}^+$.

In short, each $\gamma_{\ell,r}^+$ “portion” of $\gamma_{\ell,m}^+$ can be replaced by a product of two Hopf ring generators, one with the same or greater level and one with strictly smaller level, the latter including zero so that portion is effectively replaced by a single generator.

After equating these Hopf ring monomials coming from level- ℓ bi-partitions with outgrowth monomials, this statement implies Theorem 8.3 of [8] by applying the transfer map. Conversely, the symmetric group statement implies this up to the kernel of the transfer map, which are neutral classes.

Proof. The proof builds inductively from the Steenrod operations on Dickson algebras using the coproduct, which is the proof for symmetric groups given of Theorem 8.3 in [8]. There we use a detection system established by Madsen and Milgram [13], while here we use our detection Theorem 7.1.

We start with $m = 1$, in which case the coproducts are trivial and the restriction to $AV_{1,2^n}$ of both sides of the equality are zero, as $\ell \geq 3$. The restriction of $\gamma_{\ell,1}^+$ to V_n is the corresponding Dickson class, and the sum given by level- ℓ bipartitions yields a single possibility, the form depending on j , which coincides with that given by Hung [11] for Sq^j on the Dickson class to which it maps.

For $m > 1$, we apply Relation (9) of Theorem 8.1. Since coproduct also has a Cartan formula, we can inductively apply our present theorem. The coproduct our formula for $Sq^j \gamma_{\ell,m}^+$ agrees with Sq^j on the coproduct, as every level- ℓ bi-partition of (j', i) and (j'', k) with $j' + j'' = j$ and $i + k = m$ gives rise to a level- ℓ bipartition of (j, m) , whose corresponding term has the correct coproduct, and conversely. The restrictions to $AV_{1,2m}$ are trivial since we are taking transfer products of Hopf ring monomials where each cup monomial has at least one Hopf ring generator with $\ell \geq 3$. For $m = 2^k$ we also note that the restriction to V_n (where $n = \ell + k$) maps transfer decomposables to zero, so the restriction only depends on the bipartitions with a single term $(j, 2^k)$, and the equality again follows from compatibility with Hung’s calculation [11]. Theorem 7.1 now establishes the induction step. \square

While a corresponding statement holds for symmetric groups at all levels, for alternating groups we need modifications at the first two levels.

Definition 9.3. A modified level-2 bi-partition of (j, m) is an equality of (j, m) with a sum of the $\vec{x}_{p,\ell'}$ from Definition 9.1 for $\ell = 2$, with arbitrary non-negative coefficients, as well as $(1, 1)$ with coefficient zero or one.

Theorem 9.4. $Sq^j \gamma_{2,m}^+$ is the sum over all modified level-2 bi-partitions Π of (j, m) of Hopf ring monomials which are transfer products of $\gamma_{2+p, c_p, \ell'}^+(\pi) \gamma_{2-\ell', c_p, \ell'}^+(\pi)_{2p+\ell'}$, as well as a transfer product factor of $\gamma_{1,2}^{\circ,2}$ if $(1, 1)$ occurs non-trivially in π .

Proof. Once again we use our detection theorem and analysis of coproducts. Some exceptional behavior occurs, but propagates in a limited way for similar reasons as in the behavior for cup products given in Relation (6) of Theorem 8.1.

We treat $\gamma_{2,1}^+$ as in Theorem 5.2 and Proposition 5.4, through its restriction to the cohomology of BV_2 , namely $x_1^3 + x_1^2 x_2 + x_2^3 \in \mathbb{F}_2[x_1, x_2]^{C_3}$. While Sq^3 is forced and Sq^2 is $\gamma_{2,1}^+ \gamma_{1,2}$ as expected, $Sq^1(\gamma_{2,1}^+)$ restricts to $x_1^4 + x_1^2 x_2^2 + x_2^4$, which is the image of $(\gamma_{1,2})^2$.

As in the proof of Theorem 9.2, we complete the proof by inductively showing that the sum indicated restricts appropriately to the subgroups named in Theorem 7.1. For $m > 1$ the restriction to $AV_{1,m \cdot 2^{\ell-1}}$ of both $\gamma_{2,m}$ and all of the terms which occur in the named sum will be zero by Theorem 7.8. Restriction to V_n when $m = 2^k$ works as for $\ell > 2$, with transfer decomposables restricting to zero and single-term bipartitions giving rise to cup products of $\gamma_{\ell', 2^{k'}}$ which restrict to Hung's calculations in the corresponding Dickson algebras.

Induction is needed to check that the coproducts of the formula given for $Sq^j \gamma_{2,m}$ agree, after applying the Cartan formula for coproduct, with the tensor product of that for $Sq^i \gamma_{2,r}$ and $Sq^{i'} \gamma_{2,s}$ with $i + i' = j$ and $r + s = m$. This is mostly straightforward, except for noting that terms with a transfer-factor of $\gamma_{1,2}^2$ on both tensor factors cancel in pairs, coming from the conjugation action on the coproduct, as this term in $Sq^1 \gamma_{2,1}^+$ will also occur in $Sq^1 \gamma_{2,1}^-$. Thus there can be at most one transfer product factor of $\gamma_{1,2}^2$ on both sides of the reduced form of the coproduct, inductively implying at most one transfer product factor of $\gamma_{1,2}^2$ for any terms in $Sq^j \gamma_{2,m}$. \square

The Hopf ring generators $\gamma_{1,k;m}$ are more regular than the $\gamma_{2,m}^+$ classes in a sense, as they are all restrictions from the cohomology of symmetric groups. But our formula using level- ℓ bi-partitions or equivalently outgrowth monomials depends on transfer products, which are not preserved under the restriction map. To calculate Steenrod operations, we translate between Hopf ring presentation and presentation by cup product alone.

For example, $Sq^1(\gamma_{1,2} \odot 1_n) = \gamma_{1,1}^2 \odot \gamma_{1,1} \odot 1_n + \gamma_{2,1} \odot 1_n$. The term $\gamma_{1,1}^2 \odot \gamma_{1,1} \odot 1_n$ is then equal to $(\gamma_{1,1} \odot 1_{n+1}) \cdot (\gamma_{1,2} \odot 1_n) + \gamma_{1,3} \odot 1_{n-1}$. Thus for $m \geq 3$

$$Sq^1(\gamma_{1,2;m}^{\circ}) = \gamma_{1,1;m} \cdot \gamma_{1,2;m} + \gamma_{1,3;m} + \gamma_{2,1}^+ \odot 1_{m-2}.$$

The general case follows from classical work on the cohomology of $BO(n)$. Note that $B(C_2)^n$ maps to both $BO(n)$ and BS_{2n} with Weyl group S_n . Thus both the cohomology of $BO(n)$ and BS_{2n} map to the ring classical symmetric polynomials in n variables, the former isomorphically. Using the Cartan formula to compute Steenrod operations on $B(C_2)^n$ yields symmetric monomials which can then be translated to elementary polynomials. For example, $Sq^1(\sigma_2)$ is the symmetrization of $x_1^2 x_2$, which if there are three or more variables is equal to $\sigma_1 \sigma_2 + \sigma_3$. This shows that $Sq^1(w_2) = w_1 w_2 + w_3$, and this formula for $Sq^1(\sigma_2)$ is also the image under restriction of the calculation of $Sq^1(\gamma_{1,2} \odot 1_n)$ above.

In general Steenrod squares on $w_i \in H^*(BO(n))$, as first studied by Wu [19] and more recently Pengelley-Williams [15], will be the image under scale-one quotient of the corresponding Steenrod squares on $\gamma_{1,i} \odot 1_{n-i}$, namely $\gamma_{1,j}^2 \odot \gamma_{1,i-j} \odot 1_{m-i}$. This can be expressed as a polynomial in the $\gamma_{1,n} \odot 1_{m-n}$, which we then translate to the alternating groups setting.

Definition 9.5. Let $W(0, 0; 0) = 1^+ + 1^-$ and otherwise let

$$W(i, j; m) = \sum_{l=0}^{\min(j, n-i)} \binom{i-j+l-1}{l} \gamma_{1, j-l; m} \gamma_{1, i+l; m}.$$

If we replace all of the $\gamma_{1,k;m}$ by corresponding w_k in $W(i, j)$, we obtain Wu's formula for $Sq^j w_i$ in the cohomology of $BO(m)$. Because the scale-one subset of the cohomology of symmetric groups maps isomorphically to the same symmetric algebra as that of $BO(m)$ we obtain the following.

Theorem 9.6. $Sq^j \gamma_{1,k;m}^+$ is the sum over all level-1 bi-partitions $(j, k) = a \cdot (1, 1) + b \cdot (0, 1) + \sum c_p (2^p - 1, 2^p)$ of the transfer product of $W(a + b, a; a + b + m - k)$ with the transfer product of all γ_{1+p, c_p}^+ .

Proof. We apply the restriction map to $Sq^j(\gamma_{1,k} \odot 1_{m-k})$ in the cohomology of symmetric groups, which is the sum over level-1 bi-partitions as indicated of the transfer product of $\gamma_{1,a}^2, \gamma_{1,b}, 1_{m-k}$ and all γ_{1+p, c_p} . The transfer product of the first three is the image of $Sq^a(\gamma_{1,b} \odot 1_{m-k})$ which is then given by the Wu formula and thus restricts to $W(a + b, a; a + b + m - k)$, including when all are zero, in which case 1_0 restricts to $1^+ + 1^-$. Applying Proposition 3.6, we replace any γ_{1+p, c_p} for symmetric groups with a class that transfers to it, namely γ_{1+p, c_p}^+ , to obtain the restriction of $Sq^j(\gamma_{1,k} \odot 1_{m-k})$ as needed. \square

10. COMPONENT RINGS

Our Hopf ring presentation reduces the computationally imposing question of calculating Ext rings over alternating groups to one which can be analyzed with a rudimentary approach. Namely, our Hopf ring monomial basis is enumerable, and we can simply search for indecomposable representatives by degree, accounting for our relations, and using the Poincaré polynomial from our additive bases used to terminate the process.

In this section we give a more structured approach to the small example of \mathcal{A}_8 . As a group of order 20160, it is at the limit of current computer-based techniques. We then discuss why techniques that Feschbach developed for component rings of symmetric groups fail for alternating groups. We first develop some shorter notation, relying on the fact that the Hopf ring generators have distinct degrees once we separate the $\gamma_{1,k;m}$ classes.

Definition 10.1. • Let $\sigma_{k,m}$, or just σ_k when the component is understood, denote $\gamma_{1,k;m}$.

- Let d_i^\pm , where $i = m \cdot (2^\ell - 1)$, denote $\gamma_{\ell, m}^\pm$.
- When referring to a fixed \mathcal{A}_m , any Hopf ring monomial in σ_k and d_i^\pm is understood to define a class in its cohomology by assuming $\sigma_k = \sigma_{k,q}$ to result in full width, or taking a transfer product with a 1_q class. We drop signs from all notation for neutral classes.

For example, for \mathcal{A}_8 , $\sigma_2 = \gamma_{1,2;4}$ and $d_3 = \gamma_{2,1}^+ \odot 1_4$. Our generator names are similar to previous choices, which are natural because the σ_k restrict to elementary symmetric polynomials in the cohomology of V_1^n or $AV_{1,n}$ and the d_i (for $i > 3$) restrict to Dickson invariants on V_n .

Theorem 10.2. *The mod-two cohomology of \mathcal{A}_8 is generated as a ring under cup product by the following classes.*

Degree	2	3	4	5	6	7
Classes	σ_2	σ_3	σ_4		d_6^+	d_7^+
		d_3		$d_3 \odot \sigma_2$	d_6^-	d_7^-

Relations are:

- Products of σ_2, d_3 , or $d_3 \odot \sigma_2$ with d_7^\pm are zero (six relations).
- Products of σ_3 with $d_3, d_3 \odot \sigma_2, d_6^\pm$, or d_7^\pm are zero (six relations).
- $d_6^+ \cdot d_7^- = d_6^- \cdot d_7^+ = 0$.
- $d_7^+ \cdot d_7^- = 0$.
- $(d_3 \odot \sigma_2)^2 = \sigma_2 \cdot d_3 \cdot (d_3 \odot \sigma_2) + \sigma_2^2 \cdot (d_6^+ + d_6^-) + (d_3)^2 \cdot \sigma_4$.
- $d_6^+ \cdot d_6^- = \sigma_2 \cdot d_3 \cdot (d_3 \odot \sigma_2) + (\sigma_2)^3 \cdot (d_6^+ + d_6^-) + d_3 \cdot \sigma_4 \cdot (d_3 \odot \sigma_2) + \sigma_2 \cdot \sigma_4 \cdot ((d_3 \odot \sigma_2)^2 + d_6^+ + d_6^-)$.

Steenrod squares on generators are:

	Sq^1	Sq^2	Sq^3	Sq^4	Sq^5	Sq^6
σ_2	d_3					
σ_3		$\sigma_2\sigma_3$				
d_3	σ_2^2	$\sigma_2d_3 + d_3 \odot \sigma_2$				
σ_4	$\sigma_2 \odot d_3$	$\sigma_2\sigma_4 + d_6^+ + d_6^-$	$\sigma_3\sigma_4 + d_7^+ + d_7^-$			
$d_3 \odot \sigma_2$		$\sigma_2(d_3 \odot \sigma_2)$	$\sigma_2(d_6^+ + d_6^-)$	$\sigma_4(d_3 \odot \sigma_2)$		
d_6^\pm	$d_7^\pm + d_3\sigma_4$ $+ \sigma_2(d_3 \odot \sigma_2)$	$\sigma_2d_6^\pm$	$\sigma_4(d_3 \odot \sigma_2)$ $+ d_3d_6^\pm$	$\sigma_4d_6^\pm$ $+ (d_3 \odot \sigma_2)^2$	$(d_3 \odot \sigma_2)d_6^\pm$ $+ \sigma_4d_7^\pm$	
d_7^\pm				$\sigma_4d_7^\pm$		$d_6^\pm d_7^\pm$

Proof. The Hopf monomial basis consists of polynomials in the σ_4, d_6^+ and d_7^+ , those in σ_4, d_6^- and d_7^- , those in σ_2, σ_3 and σ_4 , and then transfer products $f_1 \odot f_2$ where the f_i are distinct polynomials in d_3^\pm and σ_2 chosen among some preferred but unnamed basis. Only the last requires an argument that it is generated by the named classes.

If f_1 and f_2 both contain a d_3^\pm then $f_1 \odot f_2 = d_6^\pm \cdot (f_1' \odot f_2')$ where f_i' are obtained by dividing by the d_3^\pm . Inductively, we need to only generate Hopf ring monomials where f_2 is σ_2 to some power. Similarly, if both f_1 and f_2 have factors of σ_2 then $f_1 \odot f_2 = \sigma_4 \cdot (f_1' \odot f_2')$ where the f_i' have one fewer factor of σ_2 . Thus it suffices to generate Hopf ring monomials of the form $d_3^i \sigma_2^j \odot 1_2$ or $d_3^i \odot \sigma_2^j$. If i is greater than one, such a monomial is a product of d_3 with the monomial with i replaced by $i - 1$ plus a term which will have a factor of d_3 on both sides and thus can be reduced. A similar argument reduces those with $j > 1$. Finally, $d_3\sigma_2 \odot 1_2 = d_3 \cdot \sigma_2 + d_3 \odot \sigma_2$, establishing our generating set.

Relations all can be verified through reducing to our Hopf ring monomial basis. That they are complete is largely a matter of observing that all of the zero products as well as the very last relation for $d_6^+ \cdot d_6^-$ leave only products of the σ_i , or those in σ_4, d_6^+ and d_7^+ or those in σ_4, d_6^- and d_7^- or those in $\sigma_2, d_3, d_3 \odot \sigma_2$ and d_6^\pm . The next-to-last relation allows us to reduce terms with $\sigma_2 \cdot d_3 \cdot (d_3 \odot \sigma_2)$ to terms where all three do not occur. Such products of $\sigma_2, d_3, d_3 \odot \sigma_2$ and d_6^\pm in which $\sigma_2 \cdot d_3 \cdot (d_3 \odot \sigma_2)$ do not occur are linearly independent. This can be seen by reducing to the Hopf monomial basis and observing that the Hopf monomial which occurs in a product with the largest constituent algebraic degree uniquely determines such a product.

The Steenrod operations are immediately verified by taking the calculations of the previous section and checking that the classes given by cup products of generators here reduce to them. \square

We compare this to the computer-generated minimal presentation on Simon King's group cohomology web-page [12]. Up to isomorphism, our choice of generators is the same through degree five; his $b_{6,0}$ and $b_{6,2}$ are our $d_6^\pm + \sigma_3^2$; and his $b_{7,6}$ and $b_{7,4}$ are our $d_7^\pm + \sigma_3 \cdot \sigma_4$. In our presentation fifteen relations – all but two – are products of two elements which vanish, while only six of the relations in the computer presentation are of this form. This cohomology was also considered by Adem, Maginnis and Milgram in [1] – see Corollary 6.5 of [2]. We have not been able to find an abstract isomorphism of their presentation with ours or King's, and suspect errors in relations between the generators and degrees 6 and 7 stemming from the V_3^\pm detection work (which inspired our own).

The last two cup product relations are fairly complicated, but are in fact simple to state in the Hopf ring monomial basis. They read as $(d_3 \odot \sigma_2)^2 = (d_3^+)^2 \odot (\sigma_2)^2$ and $d_6^+ \cdot d_6^- = (d_3)^2 \odot (\sigma_2)^3$ respectively. The Steenrod operations as given in the Hopf monomial basis instead of using cup product alone would also be simpler, especially for d_6^\pm .

Presentations using cup product alone look to be substantially more complicated starting for \mathcal{A}_{32} . To see why, we recall Feshbach's techniques for symmetric groups [5]. While he did not utilize a Hopf ring structure, his approach anticipated such. For symmetric groups, a Hopf monomial $m_1 \odot \cdots \odot m_p$ is equal to the cup product of m_p and $m_1 \odot \cdots \odot m_{p-1} \odot 1_n$ plus terms with fewer non-trivial cup monomials (or

columns, in the skyline basis). Inductively applying this column-reduction process, pure cup monomials transferred with unit classes generate the cohomology of any symmetric group. Which cup monomials are decomposable, along with relations, are determined by column reduction as well. Here we can find classes on larger symmetric groups which reduce to the single column or relation under the column reduction process, and thus agree in the indecomposables. But classes on larger symmetric groups which are too wide (that is, with more non-trivial cup monomials than can be supported on the symmetric group in question) must restrict to zero.

This basic column reduction process does not work for alternating groups and in particular for the charged classes because any class less than full width must be neutral, and the product of such must be neutral as well. Thus at the moment we only have the naive method alluded to at the beginning of this section for building from our additive basis of Hopf monomials with multiplication rules to produce generators and relations.

While finding generators and relations remains a question of interest, Hopf ring presentation has been more fruitful for applications.

For symmetric groups, the Hopf ring approach along with formulae for Steenrod operations have allowed us to calculate quotient rings and annihilator ideals for the Euler classes in the Gysin sequence, to more fully capture the relationship between Dickson algebras and cohomology of symmetric groups (which are as far from split as one could imagine, as algebras over the Steenrod algebra), and in unpublished work to decompose the Bockstein complex. The approach also was extendable to odd primes by Guerra [10], who is extending calculations to the B and D series of Coxeter groups in his PhD thesis.

For alternating groups, our presentation gives a computational proof of the lack of nilpotent elements, as the powers of an element will always yield a nonzero “tallest” Hopf ring monomial, and clearly indicates how the cohomology of the $B\mathcal{A}_{4n}$ contain interesting, manifestly unstable charged ideals.

APPENDIX A. CUP PRODUCT INPUT

For the study of symmetric groups in [8] we did not use the previously calculated cup coproduct in homology [4], which can be difficult to apply because of the need to account for Adem relations. Because Fox-Neuwirth cochains do not model cup product, contrary to what is sketched in [9], we use two such cup coproduct calculations for our study of alternating groups.

To set notation, we denote the product associated to the inclusion $\mathcal{S}_n \times \mathcal{S}_m \rightarrow \mathcal{S}_{n+m}$ by $*$, which is thus dual to our coproduct in cohomology. There are “wreath product” operations $q_i : H_k(B\mathcal{S}_n) \rightarrow H_{2k+i}(B\mathcal{S}_{2n})$ which satisfy Adem relations:

$$\text{For } m > n, \quad q_m \circ q_n = \sum_i \binom{i-n-1}{2i-m-n} q_{m+2n-2i} \circ q_i.$$

(We prefer “lower index” notation.) Given a sequence $I = i_1, \dots, i_k$ of non-negative integers, let $q_I = q_{i_1} \circ \dots \circ q_{i_k}$. Using the Adem relations, these relations are spanned by q_I whose entries are non-decreasing. We call such an I *admissible*. If such an I has no zeros we call it *strongly admissible*. Let $\iota \in H_0(B\mathcal{S}_1)$ be the non-zero class, and by abuse let q_I denote $q_I(\iota)$

Calculations of Nakaoka [14] imply that the homology of symmetric groups is a polynomial algebra over the product $*$ generated by strongly admissible q_I . Cohen-Lada-May [4] develop the theory much further, including showing that the cup coproduct is given by $\Delta_{\cup} q_I = \sum_{J+K=I} q_J \otimes q_K$. In [8] we define $\gamma_{\ell,m}$ to be the linear dual of q_I^{*m} , where I consists of ℓ ones, in the Nakaoka basis. In Theorem 4.9 of [8] we essentially show that $\gamma_{\ell,m}$ is represented by the Fox-Neuwirth cocycle with m blocks of $2^\ell - 1$ repeated ones. To verify cup product relations, we establish Fox-Neuwirth representatives of two types of products of these.

Proposition A.1. *The product $\gamma_{2,m}^2$ is represented by the Fox-Neuwirth cochain $\alpha_{2,m}(2)$ with m blocks of three repeated 2's.*

The product $\gamma_{\ell,m} \cdot \gamma_{1,m2^{\ell-1}}$ is represented by the Fox-Neuwirth cochain $\alpha_{\ell,m}(1.5)$, which has m blocks of the form $[2, 1, 2, \dots, 1, 2]$, each of length $2^\ell - 1$.

Proof. We perform the arguments in parallel. We start with $m = 1$, showing that both $\gamma_{2,1}^2$ and $[2, 2, 2]$ are linear dual in the Dyer-Lashof basis of $q_{2,2}$ (respectively, both $\gamma_{\ell,1} \cdot \gamma_{1,2^{\ell-1}}$ and $[2, 1, 2, \dots, 1, 2]$ are the linear dual of $q_{1,\dots,1,2}$). We calculate pairings, starting with $\gamma_{2,1}^2$, whose value on some x is equal to that of $\gamma_{2,1} \otimes \gamma_{2,1}$ (respectively $\gamma_{\ell,1} \otimes \gamma_{1,2^{\ell-1}}$) on Δx . For $x = q_{2,2}$ (respectively $q_{1,\dots,1,2}$) we see that the only term in $\Delta q_{2,2}$ in the correct pair of degrees is $q_{1,1} \otimes q_{1,1}$ (respectively $q_{1,\dots,1} \otimes q_{0,\dots,0,1}$), which pairs to one by definition. Any $*$ -decomposable will have decomposable coproduct, and these will evaluate to zero on a tensor factor of $\gamma_{\ell,1}$ by definition.

For pairings with $[2, 2, 2]$ (respectively $[2, 1, 2, \dots, 1, 2]$), $*$ -decomposable classes also evaluate to zero, by the analogue of Theorem 4.16. We then apply the symmetric groups version of Proposition 4.15 to evaluate $[2, 2, 2]$ on $q_{2,2}$, which by definition is represented by a map from $S^2 \times_{S_2} (\mathbb{R}P^2 \times \mathbb{R}P^2)$ to $\overline{\text{Conf}}_4(\mathbb{R}^\infty)$ sending $(u, v, w)/\sim$ to the configuration with points at $u \pm \varepsilon v$ and $-u \pm \varepsilon w$. We get a single transversal intersection when u, v and w are all the equivalence class of $(0, 0, 1)$. (To evaluate $[2, 1, 2, \dots, 1, 2]$ on $q_{1,\dots,1,2}$ we represent the latter by a standard map of the iterated wreath product $S^1 \int \dots \int \mathbb{R}P^2$, where $S^1 \int X = S^1 \times_{S_2} (X \times X)$, to $\overline{\text{Conf}}_{2^\ell}(\mathbb{R}^\infty)$. Once again, we will get a single transversal intersection when each coordinate entry of S^1 is $(0, \pm 1)$ and each of $\mathbb{R}P^2$ is the equivalence class of $(0, 0, 1)$.)

To pass to higher m , we perform an induction based on the detection result of Madsen and Milgram [13], that the cohomology of symmetric groups is detected by coproduct, along with restriction to V_n for S_{2^n} . The formulae for coproducts of $\gamma_{2,m}^2$ and $\alpha_{2,m}(2)$ (respectively $\gamma_{\ell,m} \cdot \gamma_{1,m2^{\ell-1}}$ and $\alpha_{\ell,m}(1.5)$), the former given by Theorem 3.1 and Hopf ring distributivity and the latter given by Theorem 4.16, will be equal by inductive assumption, and application of detection establishes the induction step when $m \neq 2^n$. When $m = 2^n$ we note that by having the same coproducts they must agree up to the kernel of restriction to V_n , which is generated by $\gamma_{n,1}$. But there are no non-zero multiples of $\gamma_{n,1}$ in the relevant degrees. \square

The cup product representatives thus far have been obtained simply by adding Fox-Neuwirth entries of the factors. In [9] we define a Hopf ring structure on Fox-Neuwirth cochains in this way, the homology of which agrees with the cohomology of symmetric groups. But, contrary to what is claimed and sketched in [9], this abstract isomorphism is not induced by the map between them given by Alexander duality as in the proof of Theorem 4.9. For example, while $\gamma_{1,2}$ is represented by $[1, 0, 1]$, its cube $\gamma_{1,2}^3$ cannot be represented by $[3, 0, 3]$. Its cube evaluates non-trivially with the class $q_{2,2}$ in homology, whose coproduct includes $q_{2,0} \otimes q_{0,2} = q_{0,1} \otimes q_{0,2}$ by the first Adem relation. But the image of $q_{2,2}$ can be embedded in configurations in \mathbb{R}^3 and so cannot pair with $[3, 0, 3]$. Indeed, $\gamma_{1,2}^3$ must be represented by $[3, 0, 3] + [2, 2, 2]$. There are variants of Fox-Neuwirth cochains which should result in cup product models as well. Thankfully, our need for cup product input at the cochain level was limited in the present work.

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