# OPEN QUESTIONS ON THE COHOMOLOGY OF SYMMETRIC GROUPS AND RELATED TOPICS

- Calculate the pairing between the Kudo-Araki-Dyer-Lashof basis for homology and skyline basis for cohomology of symmetric groups. The key case is the pairing between the span of $q_I$ and the $\odot$-indecomposibles on $B\mathcal{S}_{2^n}$.

  This pairing can be calculated in cases in two ways, either through using the cup coproduct formula in the Kudo-Araki-Dyer-Lashof algebra ($\Delta q_I = \sum_{J+K=I} q_J \otimes q_K$ or by using the composition coproduct in the dual of this algebra as calculated by Pengelley and Williams in their "Sheared algebra maps..." paper.

  Using the standard relationship between the Steenrod and Kudo-Araki-Dyer-Lashof algebras, compare the skyline basis (monomial basis for $R_k$) with the Milnor basis.

- Recall the Yoneda approach to Ext groups through extensions of modules. For example, $\mathbb{F}_2 \to sgn \to \mathbb{F}_2$ represents the class $\gamma_1 \odot 1_{n-2}$ in $H^1(B\mathcal{S}_n)$. Find extensions which represent the skyline (Hopf monomial) basis, in particular the Hopf ring generators. Some interesting $\mathbb{F}_2$-representations of $\mathcal{S}_n$ could appear.

- As discussed in the lecture, understand tensor and exterior products of Dickson algebras. Both understand topological significance of and calculate the extensions of these which occur in the cohomology of symmetric groups.

- Following the resolution of the Segal Conjecture by Carlsson, mathematician such as Bob Oliver applied the result to give stable splittings of $BG$ for finite $G$. The question is to understand the corresponding splittings for cohomology of symmetric groups as modules over the Steenrod algebra. A first look at $B\mathcal{S}_4$ shows that the skyline basis is not the right basis for splitting - one needs linear combinations of basis elements (once again because terms occurring in Steenrod operations can have further transfer product operations).

- * Verify the conjecture on the Margolis homology of Dickson algebras, namely that the $Q_i$ homology of $D_n$ when $i \geq n$ is $\mathbb{F}_2[d_{k,\ell}^2]/\left(Q_i(d_{0,n}), Q_i(d_{k,\ell} \cdot d_{0,n})\right)$, as the $d_{k,\ell}$ ranges over Dickson generators.

- * Investigate the commutative algebra, in particular the prime spectrum, of $H^*(B\mathcal{S}_n)$. Use the prime spectrum to make explicit computations of support varieties, and work with representation theorests (if you aren't one yourself) to understand their significance and connections to other approaches. Support varieties of Specht moldules for example would be of interest. ("Recall" that the support variety of a $G$-module $M$ is the variety associated to the kernel of the map from $\mathrm{Ext}_{k[G]}(k,k)$ to $\mathrm{Ext}_{k[G]}(M,M)$. One can imagine other invariants of $M$ in a similar vein.)

- Present the cohomology of $B\mathcal{S}_n$ (including/ starting with? $n = \infty$) as an $\mathcal{A}$-algebra.

- Expository: give a proof of Nishida's Nilpotence Theorem which instead of only using the Nishida relations for dual Steenrod operations on homology of symmetric groups uses Steenrod operations on cohomology in addition or instead.

- Develop algebraic geometry of Hopf rings, and understand its interplay with component rings in the component setting.

- Find ring generators for cohomology of alternating groups.

- Calculate cohomology rings of symmetric groups with integer coefficients. The Bockstein homology has been determined (conjecturally, and not written up).

- Find direct sums of $\mathrm{Ext}_{k[\mathcal{S}_n]}(M, N)$ which support Hopf ring structure extending that for trivial coefficients and try to make calculations (which are likely quite difficult).

- The Galatius-Madsen-Tillman-Weiss theorem identifies $B\Gamma_\infty$, where $\Gamma_\infty$ is the stable mapping class group, with $\Omega^\infty MTSO(2)$. The de-looping of this sits in a fiber sequence of infinite loop spaces
$$\Omega^\infty \Sigma MTSO(2) \to Q\Sigma\mathbb{C}P^\infty_+ \to QS^0,$$
which Galatius used to compute homology. Make such calculations, in particular ring structure, more explicit using calculations of cohomology ring structure of $QX$. (Galatius mention that his calculations were limited by lack of control of cup coproduct in the Kudo-Araki-Dyer-Lashof algebra.)

- Revisit Wellington's calculations of the unstable Adams spectral sequence for $QS^0$ (another area which was limited by knowledge of cup coproduct on homology of symmetric groups). (You may want to start by presenting the cohomology of $S_n$ as an $\mathcal{A}$-algebra as mentioned above.)

- Understand the topological significance of and make calculations of the extensions of tensor products/ exterior products of Dickson algebras which occur in the transfer product filtration.

- Extend the geometry of $H^*(B\mathcal{S}_\infty)$ to $H^*(QS^0)$ as "characteristic classes". These are isomorphic, but only the former has geometric representatives, namely by embedding a finite cover of $X$ in some $X \times \mathbb{R}^n$ and then looking at where dyadic collections share coordinates. If instead of a finite cover of $X$ (which is what one can pull back from $X \to B\mathcal{S}_N$) one takes $f : X \to \Omega^N S^N$ and considers the adjoint $\Sigma^N X \to S^N$, what geometry can one investigate for this adoint to describe the result in cohomology of $f$?

Talk to me if you're interested in any of these!