Cohomology of symmetric groups

Dev P. Sinha
University of Oregon

Joint work with Chad Giusti, Lorenzo Guerra, David Pengelley and Paolo Salvatore.

Cảm ơn bạn!

Group cohomology

Recall that there is a unique homotopy type obtained by quotienting any free contractible G-space by its G-action. By abuse call any representative BG.

It serves as a bridge between topology and algebra. For finite G,

$$H^*(BG;k) \cong \mathrm{Ext}_{k[G]}(k;k). \quad \text{(Almost by definition)}$$

This is the ground ring for Borel equivariant cohomology. Closely related: the Borel(-Serre) spectral sequence $H^*(BG;\mathcal{H}^*(X)) \implies H^*(X/G)$ for a free G action on X.

Group cohomology

BG is important in algebra, not only because its cohomology is *Ext* over $k[G]$. For example:

$$K^0(BG) \cong \mathrm{Rep}_{\mathbb{C}}(G)_{\hat{I}}. \quad \text{(Atiyah-Segal)}$$

So the Atiyah-Hirzebruch spectral sequence provides a connection between representation theory and group cohomology.

A new contribution: the "support variety" in $H^*(BG; k)$ of its action on $\mathrm{Ext}_{k[G]}(M; M)$ is a rich invariant of a G-representation M.

Despite being basic, relatively few explicit calculations of group cohomology rings have been made, a large percentage being in Adem & Milgram's book or by machine.

Symmetric groups

Symmetric groups are privileged among groups, acting on iterated products and coproducts, and related constructions such as configuration spaces.

Concomitantly, symmetric groups play special roles in topology.

- They play a central role in the definition of Steenrod operations, through symmetric group actions on products.
- Barratt-Priddy-Quillen-Segal: $H^*(\Omega^\infty \Sigma^\infty S^0) \cong H^*(B\mathcal{S}_\infty)$.
- The Goodwillie derivatives of a functor are infinite-loop spaces representing homotopy orbits of $\mathcal{S}_n$-equivariant spectra.

Homology and cohomology of symmetric groups

The ranks of homology groups of BS_n has been understood since work of Kudo-Araki, Dyer-Lashof, Nakaoka, Cohen-Lada-May.

The ring structure of the cohomology of BS_∞ at the prime two was calculated by Nakaoka in 1960.

The coproduct structure for $H_*(B\mathcal{S}_n)$ was given by Cohen-Lada-May. As we will see, it is difficult to work with because of a need to apply Adem relations.

Through the '80s and '90s, Hưng, Adem, Milgram, McGinnis and Feshbach studied individual symmetric groups to better understand cohomology ring and Steenrod structure.

Homology and cohomology of symmetric groups

Theorem 5.4

$$H^*(\mathcal{S}_{12}) \cong P[\sigma_1, \sigma_2, \sigma_3, c_3, \sigma_4, \sigma_5, \sigma_6, d_6, d_7, d_9](x_5, x_7, x_8)/\langle R\rangle$$

where deg $\sigma_i = i$, deg $c_3 = 3$, deg $d_i = i$, deg $x_i = i$ *and* R *is the set of relations*

$$x_7^2 + \sigma_4 x_5^2 + \sigma_2\sigma_4 x_8 + (\sigma_6 c_3 + \sigma_2\sigma_4 c_3)\, x_5 + \sigma_4^2 + c_3^2 + \sigma_2^2\sigma_4 d_6 + \sigma_6\sigma_2 d_6,$$
$$x_8^2 + d_6 x_5^2 + c_3 d_6 x_7 + (d_9\sigma_2 + c_3 d_6 \sigma_2) x_5 + d_6^2\sigma_2^2 + c_3^2\sigma_4 d_6 + d_9 c_3 \sigma_2,$$
$$x_5 x_7 + \sigma_2 x_5^2 + \left[\sigma_2^2 + \sigma_4\right] x_8 + \sigma_2^2 c_3 x_5 + \sigma_2^3 d_6 + \sigma_6 c_3^2 + \sigma_6 d_6 + \sigma_2\sigma_4 c_3^2,$$
$$x_5 x_8 + c_3 x_5^2 + \left[c_3^2 + d_6\right] x_7 + c_3^2\sigma_2 x_5 + c_3^3\sigma_4 + d_9\sigma_2^2 + d_9\sigma_4 + c_3 d_6 \sigma_2^2,$$
$$x_5^3 + c_3\sigma_2 x_5^2 + x_7 x_8 + c_3\sigma_2^2 x_8 + \sigma_2 c_3^2 x_7 + \left(\sigma_2^2 d_6 + \sigma_4 c_3^2\right) x_5 + \sigma_2^3 d_9 + c_3^3\sigma_6 + \sigma_6 d_9,$$

$$d_9\sigma_1, d_9\sigma_3, d_9\sigma_5$$
$$d_7\sigma_3, d_7 x_5, d_7\sigma_1 c_3, d_7(x_7 + \sigma_4 c_3), d_7(\sigma_5 + \sigma_4\sigma_1), d_7(\sigma_6 + \sigma_4\sigma_2),$$
$$d_7(x_8 + d_6\sigma_2), d_7(d_9 + d_6 c_3),$$

$$x_7\sigma_1 + x_5(\sigma_1\sigma_2 + \sigma_3) + c_3\left(\sigma_2\sigma_3 + \sigma_2^2\sigma_1 + \sigma_1\sigma_4 + \sigma_5\right),$$
$$x_7\sigma_3 + x_5(\sigma_5 + \sigma_1\sigma_4) + c_3(\sigma_1\sigma_6 + \sigma_3\sigma_4 + \sigma_1\sigma_2\sigma_4),$$
$$x_7\sigma_5 + x_5\sigma_1\sigma_6 + c_3(\sigma_3\sigma_6 + \sigma_1\sigma_2\sigma_6),$$
$$x_8\sigma_1 + d_6(\sigma_3 + \sigma_1\sigma_2),$$
$$x_8\sigma_3 + d_6(\sigma_5 + \sigma_1\sigma_4),$$
$$x_8\sigma_5 + d_6\sigma_1\sigma_6 .$$

First main result

Theorem (Giusti-Salvatore-S)

The direct sum of cohomology $\bigoplus_n H^*(B\mathcal{S}_n; \mathbb{F}_2)$ *is a free divided-powers component Hopf ring primitively generated by classes* $\gamma_\ell \in H^{2^\ell - 1}(B\mathcal{S}_{2^\ell})$.

All of the information from the previous slide – along with the increasingly complex lists for larger symmetric groups – follows from this statement.

We also explicitly treat Steenrod structure, building on a calculation of Hưng.

Plan

1. Lecture 1 (from here) - setting the stage, through configuration models and Hopf ring structure.
2. Lecture 2 - elaboration of the main theorem, the homology of symmetric groups, and some proofs.
3. Lecture 3 - restriction to subgroups, Steenrod structure, and $B\mathcal{S}_\infty$.
4. Lecture 4 - cohomology of alternating groups and Fox-Neuwirth resolutions.
5. Lecture 5 - odd primes, divided powers and free infinite loop spaces.
6. Lecture 6 - Margolis homology; vistas.

Configuration models

Let $\mathrm{Conf}_n(\mathbb{R}^m)$ denote the configuration space of $n$distinct ordered points in $\mathbb{R}^m$ - that is the subspace of $(x_1, \ldots, x_n)$ where $x_i \neq x_j$ when $i \neq j$.

We denote by $\overline{\mathrm{Conf}}_n(\mathbb{R}^m)$ the quotient $\mathrm{Conf}_n(\mathbb{R}^m)/\mathcal{S}_n$, which is the space of unlabeled configurations of n points in $\mathbb{R}^m$.

Exercise: show that $\overline{\mathrm{Conf}}_n(\mathbb{R}^\infty)$ is a model for $B\mathcal{S}_n$.

Configuration models

Familiar group-theoretic constructions have geometric representatives.

The inclusion $\mathcal{S}_n \times \mathcal{S}_m \hookrightarrow \mathcal{S}_{n+m}$ is given by "'stacking configurations next to each other."

This makes $\coprod B\mathcal{S}_n = \coprod \overline{\mathrm{Conf}}_n(\mathbb{R}^\infty)$ an H-space (in fact, A_∞).

Configuration models

Homology classes can represented by closed submanifolds (through their fundamental classes) and cohomology classes represented by proper submanifolds (through intersection/ Poincar'e duality).
For example, $\overline{\mathrm{Conf}}_2(\mathbb{R}^\infty) \simeq \mathbb{R}P^\infty$.

Its d-dimensional homology is represented by pairs of antipodal points around some $S^d \subset \mathbb{R}^\infty$.
Its d-dimensional cohomology is represented by pairs of points which say share their first d-coordinates.

We will see that all mod-two homology and cohomology of symmetric groups can be represented as such.

Hopf ring structure

Definition

A Hopf ring is a ring object in the category of coalgebras.

Explicitly, a Hopf ring is vector space V with two multiplications, one comultiplication, and an antipode $(\odot, \cdot, \Delta, S)$ such that the first multiplication forms a Hopf algebra with the comultiplication and antipode, the second multiplication forms a bialgebra with the comultiplication, and these structures satisfy the distributivity relation

$$\alpha \cdot (\beta \odot \gamma) = \sum_{\Delta\alpha = \sum a' \otimes a''} (a' \cdot \beta) \odot (a'' \cdot \gamma).$$

Hopf ring structure

A free "algebra" on two products would be very large, but the distributivity relation cuts things down considerably.

$$\alpha \cdot (\beta \odot \gamma) = \sum_{\Delta\alpha = \sum a' \otimes a''} (a' \cdot \beta) \odot (a'' \cdot \gamma).$$

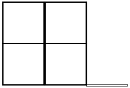
Consequence: every element can be reduced to Hopf monomials $m_1 \odot m_2 \odot \cdots \odot m_i$, where the m_i are monomials in the $\cdot$-product alone.

A Hopf ring example

We may use Young diagrams to represent symmetric polynomials - that is elements of $R[x_1, \cdots, x_n]^{\mathcal{S}_n}$ - through symmetrized monomials.

$$P \leftrightarrow \mathrm{Sym}(x^P),$$

where $\mathrm{Sym}(m)$ denotes the minimal symmetrization of a monomial m, namely $\sum_{[\sigma] \in S_n/H} \sigma \cdot m$ where H fixes m.



$$\leftrightarrow {x_1}^2 {x_2}^2 + {x_1}^2 {x_3}^2 + {x_2}^2 {x_3}^2.$$

A diagrammatic approach to invariant theory

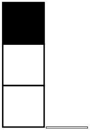
Multiplication is given by the sum of “stackings” over all possible column matchings *up to automorphism*.

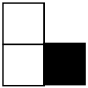
Example: the previous monomial times the “1-block.”

This seems ineffective - one doesn't immediately see the fact that these are polynomial rings!

A diagrammatic approach to invariant theory

But doing this with colored blocks is a fruitful approach to invariants of multiple sets of variables.

 $\leftrightarrow {x_1}^2 y_1 + {x_2}^2 y_2.$

 $\leftrightarrow {x_1}^2 y_2 + {x_2}^2 y_1.$

A diagrammatic approach to invariant theory

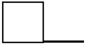
These diagrams immediately define additive bases for these rings of invariants, with multiplication still represented by stacking.

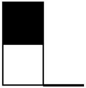
On the other hand, presenting these rings in positive characteristic by generators and relations over has been notoriously difficult.

In particular, Feshbach gave explicit generators and inductively defined relations for these rings over $\mathbb{F}_2$, and to our knowledge such descriptions are still open for these rings over $\mathbb{F}_p$ with $p > 2$.

A diagrammatic approach to invariant theory

A set of generators for $\mathbb{F}_2[x_1, x_2, y_1, y_2]^{\mathcal{S}_2}$ is

$a =$ , $b =$ , $c =$ , $d =$ 

$e =$ , $f =$ ,

With relations:

$$ab = e + f$$

and

$$ef = a^2 d + b^2 c.$$

Hopf ring structure

Direct sums of rings of symmetric invariants form Hopf rings.

The $\cdot$ multiplication is the standard one, defined to be zero if the number of variables does not agree.

The $\odot$ multiplication is an induction product: "reindex and symmetrize."

The comultiplication is defined by "restricting" to get products of symmetric invariants of lower numbers of variables.

Hopf ring structure

Definition

A divided powers Hopf ring generated by a finite set $a_1, \cdots a_k$ is the Hopf ring generated under the two products by variables $a_{i[n]}$ with $1 \leq i \leq k$ and $n \geq 1$ with coproducts determined by

$$\Delta a_{i[n]} = \sum_{k+\ell=n} a_{i[k]} \otimes a_{i[\ell]},$$

and $\odot$-products

$$a_{i[n]} \odot a_{i[m]} = \binom{n+m}{n} a_{i[n+m]}.$$

Hopf ring structure

Theorem (GSS)

The direct sum of rings of invariants in k collections of variables (for example, $\bigoplus_n A[x_1, \cdots, x_n, y_1, \cdots, y_n]^{\mathcal{S}_n}$ for $k = 2$) is a free divided powers $\mathbb{N}$-component Hopf ring, with k generators on the first component, given by the variables themselves.

This is a highly relevant example: restriction maps from cohomology of symmetric groups to that of their elementary abelian subgroups respects Hopf ring structure and send generators to generators.

Hopf ring structure

There are similar Hopf ring structures on representation rings of symmetric groups, defined by tensor product, induction product, and restriction coproduct. The induction/restriction Hopf algebra structure was studied by Zelevinsky.
Strickland and Turner first saw these structures in (generalized) group cohomology.

Theorem (Strickland-Turner)

For any ring-theory E^*, the cohomology of symmetric groups $\bigoplus_n E^*(B\mathcal{S}_n)$ forms a (derived) Hopf ring where

- *The $\cdot$ product is the standard (cup) product (with zero products between distinct summands).*
- *The coproduct Δ is induced by the standard covering $p : B\mathcal{S}_n \times B\mathcal{S}_m \to B\mathcal{S}_{n+m}$.*
- *The product $\odot$ is the transfer associated to p.*

Hopf ring structure and geometry

Tying back to our geometric model $\overline{\mathrm{Conf}}_n(\mathbb{R}^\infty)$ for $B\mathcal{S}_n$, recall that cohomology classes are defined by "conditions on points" such as "two points sharing d coordinates".

And as we will see others such as "four points sharing d coordinates,"

and even "eight points sharing m coordinates, another eight points sharing n coordinates and those eight points also break up into four sets of two which share k coordinates."

Hopf ring structure and geometry

For such representatives, these Hopf ring structures are interpreted as follows

- The $\cdot$ product is the cup product, which as always is defined by intersection, which means imposing both conditions (simultaneously, on the same collection of points, and keeping in mind the configurations are unordered).
- The coproduct Δ means restricting conditions to configurations which have been "stacked".
- The product $\odot$ means taking conditions on m and n points and getting a condition on $m + n$ points by finding disjoint subsets which satisfy those conditions.

This is how these structures were re-discovered.

Summary

The cohomology of symmetric groups is a central topic. Their classifying spaces have pleasing geometric representatives, which can be used to better understand their structure.

There is a long history of calculations, with ring structures complicated to describe.

Ring structures are much simpler to describe when one considers all symmetric groups together, in which case the cohomology constitutes a Hopf ring incorporating the cup product. We see similar phenomena for rings of invariants and representation rings of symmetric groups.