Cohomology of symmetric groups

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First main result

Theorem (Giusti-Salvatore-S)

The direct sum of cohomology $\bigoplus_n H^*(B\mathcal{S}_n; \mathbb{F}_2)$ *is a free divided-powers component Hopf ring primitively generated by classes* $\gamma_\ell \in H^{2^\ell - 1}(B\mathcal{S}_{2^\ell})$.

Our first goal today is to "unpack" this to see how to make calculations, seeing this leads to a preferred additive basis and simple rules to follow to multiply.

Hopf ring structure

Definition

A Hopf ring is a ring object in the category of coalgebras.

Explicitly, a Hopf ring is vector space V with two multiplications $\odot, \cdot$, one comultiplication, Δ and an antipode S such that the first multiplication forms a Hopf algebra with the comultiplication and antipode, the second multiplication forms a bialgebra, and these structures satisfy the distributivity relation

$$\alpha \cdot (\beta \odot \gamma) = \sum_{\Delta\alpha = \sum a' \otimes a''} (a' \cdot \beta) \odot (a'' \cdot \gamma).$$

Every element can be reduced to Hopf monomials $m_1 \odot m_2 \odot \cdots \odot m_i$, where the m_i are monomials in the $\cdot$-product alone.

Component divided powers Hopf ring structure

A component Hopf ring decomposes as $\bigoplus_n R_n$, where the $\cdot$ is zero between summands. This is immediate for $\bigoplus_n H^*(B\mathcal{S}_n)$.

The divided powers structure is for the "plus product" $\odot$, giving operations $H^*(B\mathcal{S}_m) \to H^*(B\mathcal{S}_{km})$. We typically denote it $x \mapsto x_{[k]}$ (reserving superscripts for exponentiation of $\cdot$). Recall that divided powers over $\mathbb{F}_2$ is exterior, so $x \odot x = 0$ but we see this as $x \odot x = 2x_{[2]}$.

The divided powers operation commutes with $\cdot$ on Δ-primitives, and satisfies the coproduct formula $\Delta x_{[n]} = \sum_{i+j=n} x_{[i]} \otimes x_{[j]}$.

Component divided powers Hopf ring structure

The divided powers operation is defined through the map on classifying spaces given by the inclusion of the wreath product $\mathcal{S}_m \wr \mathcal{S}_k$ in $\mathcal{S}_{mk}$.

A cohomology class $x \in H^* B\mathcal{S}_m$ gives rise to a cohomology class $x^{\otimes} m \in H^*(B\mathcal{S}_m \wr \mathcal{S}_k)$, to which we then apply the transfer.

In the geometry of configuration spaces it "repeats k times the conditions which define x."

Graphical representation

As we did for rings of invariants, we translate from algebraic notation to graphical representatives.

Generators γ_ℓ are represented by blocks. We give them width $2^{\ell-1}$, so that the width corresponds to half the component.
We choose height so that area is equal to degree, namely $2^\ell - 1$.

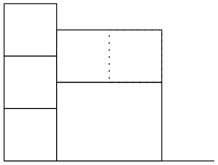
The cup $\cdot$ product is represented by stacking vertically, and the transfer $\odot$ product is represented by stacking horizontally. Divided power is denoted by repeated stacking horizontally with dashed lines.

The unit class 1_m on the $B\mathcal{S}_m$ component is indicated by an "empty space."

Graphical representation

$$\gamma_1^3 \odot \gamma_{1[2]} \cdot \gamma_2 \odot 1_2$$

is represented by:



These graphical representations for Hopf ring monomials are called "skyline diagrams."

Additive basis

To form an additive basis, inductively:

- Take transfer products (without repeats) of basis elements for previous symmetric groups. This can be done systematically so as to not have duplicates.
- On 2^ℓ components, take all $\cdot$ products of $\gamma_{i[2^j]}$ with $i + j = \ell$ (that is, a basis for the polynomial algebra on these).

These latter form a basis for the $\odot$-indecompsibles of this Hopf ring, from which everything else is built through $\odot$. We will view these important subrings through a few lenses, in particular identifying them with Dickson algebras.

Hopf ring generators

	$B\mathcal{S}_2$	$B\mathcal{S}_4$	$B\mathcal{S}_6$	$B\mathcal{S}_8$	$B\mathcal{S}_{10}$	$B\mathcal{S}_{12}$	$B\mathcal{S}_{14}$	$B\mathcal{S}_{16}$
15								γ_4
14								$\gamma_{3[2]}$
13								
12								$\gamma_{2[4]}$
11								
10								
9						$\gamma_{2[3]}$		
8								$\gamma_{1[8]}$
7				γ_3			$\gamma_{1[7]}$	
6				$\gamma_{2[2]}$		$\gamma_{1[6]}$		
5					$\gamma_{1[5]}$			
4				$\gamma_{1[4]}$				
3		γ_2	$\gamma_{1[3]}$					
2		$\gamma_{1[2]}$						
1	γ_1							

Multiplication

The main payoff of our techniques are much better control of mulitplication, as well as new insight into Steenrod structure.

Multiplication requires repeated use of Hopf ring distributivity, though we will be able to streamline things graphically.

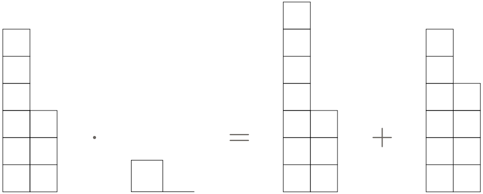
$$
\begin{aligned}
(\gamma_{1[2]} \odot 1_2) \cdot (\gamma_{1[2]}^2 \odot 1_2) &= \sum_{\Delta \gamma_{1[2]} \odot 1_2 = \sum a_i \otimes b_i} (a_i \cdot \gamma_{1[2]}^2) \odot (b_i \cdot 1_2) \\
&= (\gamma_{1[2]} \cdot \gamma_{1[2]}^2) \odot (1_2 \cdot 1_2) + (\gamma_1 \odot 1_2 \cdot \gamma_{1[2]}^2) \odot (\gamma_1 \cdot 1_2) \\
&= \gamma_{1[2]}^3 \odot 1_2 + \gamma_1^3 \odot \gamma_1^2 \odot \gamma_1
\end{aligned}
$$

Multiplication

Graphically, transfer product corresponds to placing two column Skyline diagrams next to each other and merging columns with the same constituent blocks, with a coefficient of zero if any of those column widths share a one in their dyadic expansion.

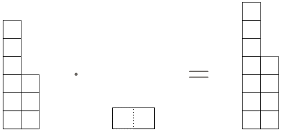
For cup product, we start with two column diagrams and consider all possible ways to split each into columns, along either original boundaries of columns or along the vertical lines of full height internal to the rectangles representing $\gamma_{\ell,\boldsymbol{n}}$. We then match columns of each in all possible ways up to automorphism, and stack the resulting matched columns to get a new set of columns.

Multiplication

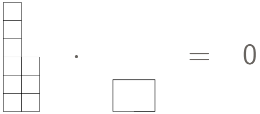


$$(\gamma_1^i \odot \gamma_1^j) \cdot (\gamma_1 \odot 1_2) = \gamma_1^{i+1} \odot \gamma_1^j + \gamma_1^i \odot \gamma_1^{j+1}$$

Multiplication

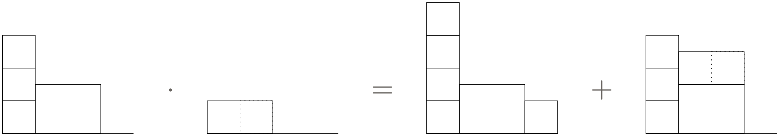


$$(\gamma_1^i \odot \gamma_1^j) \cdot \gamma_{1[2]} = \gamma_1^{i+1} \odot \gamma_1^{j+1}$$



$$(\gamma_1^i \odot \gamma_1^j) \cdot \gamma_2 = 0$$

Multiplication



$$(\gamma_1^3 \odot \gamma_2 \odot 1_2) \cdot (\gamma_{1[2]} \odot 1_4) \quad = \quad \gamma_1^4 \odot \gamma_2 \odot 1_2 + \gamma_1^3 \odot \gamma_{1[2]}\gamma_2 \odot 1_2$$

Exercises

Calculate products such as

- $(\gamma_1 \odot 1_2) \cdot \gamma_{1[2]}^2$
- $(\gamma_1 \odot 1_2) \cdot (\gamma_1^2 \odot \gamma_1)$
- $(\gamma_1 \odot 1_2) \cdot \gamma_2$
- $(\gamma_1^3 \odot \gamma_2 \odot 1_2) \cdot (\gamma_{1[2]} \odot 1_4)$.

Or give a basis of $B\mathcal{S}_{12}$

Or ponder this presentation as you like.

Talking with your neighbors encouraged.

Ring presentations and beyond

If one wants generators and relations for an individual symmetric groups, finding such from an additive basis with multiplication rules is readily algorithmic, by proceeding by degree.

Such are necessarily complicated; the Adem-Mcginnis-Milgram presentation for $B\mathcal{S}_{12}$ is minimal, for example.

We recover Feshbach's generators. His relations are inductively defined - it would be nice to have a "closed form."

There are many other open questions, starting with for example computing the prime ideal spectrum.

Ring presentations and beyond

Theorem 5.4

$$H^*(\mathcal{S}_{12}) \cong P[\sigma_1, \sigma_2, \sigma_3, c_3, \sigma_4, \sigma_5, \sigma_6, d_6, d_7, d_9](x_5, x_7, x_8)/\langle R\rangle$$

where deg $\sigma_i = i$, deg $c_3 = 3$, deg $d_i = i$, deg $x_i = i$ *and* R *is the set of relations*

$$\begin{aligned}
&x_7^2 + \sigma_4 x_5^2 + \sigma_2\sigma_4 x_8 + (\sigma_6 c_3 + \sigma_2\sigma_4 c_3)\, x_5 + \sigma_4^2 + c_3^2 + \sigma_2^2\sigma_4 d_6 + \sigma_6\sigma_2 d_6,\\
&x_8^2 + d_6 x_5^2 + c_3 d_6 x_7 + (d_9\sigma_2 + c_3 d_6 \sigma_2) x_5 + d_6^2\sigma_2^2 + c_3^2\sigma_4 d_6 + d_9 c_3 \sigma_2,\\
&x_5 x_7 + \sigma_2 x_5^2 + \left[\sigma_2^2 + \sigma_4\right] x_8 + \sigma_2^2 c_3 x_5 + \sigma_2^3 d_6 + \sigma_6 c_3^2 + \sigma_6 d_6 + \sigma_2\sigma_4 c_3^2,\\
&x_5 x_8 + c_3 x_5^2 + \left[c_3^2 + d_6\right] x_7 + c_3^2\sigma_2 x_5 + c_3^3\sigma_4 + d_9\sigma_2^2 + d_9\sigma_4 + c_3 d_6 \sigma_2^2,\\
&x_5^3 + c_3\sigma_2 x_5^2 + x_7 x_8 + c_3\sigma_2^2 x_8 + \sigma_2 c_3^2 x_7 + \left(\sigma_2^2 d_6 + \sigma_4 c_3^2\right) x_5\\
&\quad + \sigma_2^3 d_9 + c_3^3 \sigma_6 + \sigma_6 d_9,
\end{aligned}$$

$$\begin{aligned}
&d_9\sigma_1, d_9\sigma_3, d_9\sigma_5\\
&d_7\sigma_3, d_7 x_5, d_7\sigma_1 c_3, d_7(x_7 + \sigma_4 c_3), d_7(\sigma_5 + \sigma_4\sigma_1), d_7(\sigma_6 + \sigma_4\sigma_2),\\
&d_7\left(x_8 + d_6\sigma_2\right), d_7\left(d_9 + d_6 c_3\right),
\end{aligned}$$

$$\begin{aligned}
&x_7\sigma_1 + x_5\left(\sigma_1\sigma_2 + \sigma_3\right) + c_3\left(\sigma_2\sigma_3 + \sigma_2^2\sigma_1 + \sigma_1\sigma_4 + \sigma_5\right),\\
&x_7\sigma_3 + x_5\left(\sigma_5 + \sigma_1\sigma_4\right) + c_3\left(\sigma_1\sigma_6 + \sigma_3\sigma_4 + \sigma_1\sigma_2\sigma_4\right),\\
&x_7\sigma_5 + x_5\sigma_1\sigma_6 + c_3\left(\sigma_3\sigma_6 + \sigma_1\sigma_2\sigma_6\right),\\
&x_8\sigma_1 + d_6\left(\sigma_3 + \sigma_1\sigma_2\right),\\
&x_8\sigma_3 + d_6\left(\sigma_5 + \sigma_1\sigma_4\right),\\
&x_8\sigma_5 + d_6\sigma_1\sigma_6\ .
\end{aligned}$$

Homology of symmetric groups

Our proofs build on the well-understood homology of symmetric groups.

Recall that we prefer the $\overline{\mathrm{Conf}}_n(\mathbb{I}^\infty)$ model for $B\mathcal{S}_n$. (Where $\mathbb{I} = [-1, 1]$, so we are switching to configurations in a "box".)

In this model, the product $B\mathcal{S}_n \times B\mathcal{S}_m \to B\mathcal{S}_{n+m}$ is defined by "stacking next to."

Explicitly, if $f : M \to \overline{\mathrm{Conf}}_i(\mathbb{I}^\infty)$ and $g : N\overline{\mathrm{Conf}}_j(\mathbb{I}^\infty)$ represent homology classes x and y then $x * y$ is represented by a map $M \times N \to \overline{\mathrm{Conf}}_{i+j}(\mathbb{I}^\infty)$ sending (m, n) to

$$\frac{1}{3}(f(m) - v) \bigsqcup \frac{1}{3}(g(n) + v),$$

for a fixed unit vector v.

Homology of symmetric groups

There is a choice of in which direction one stacks. By making continuous families of such choices, in the case of the product of a class with itself, we get "higher products" or operations.

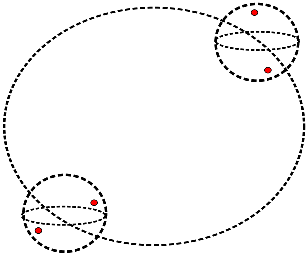
Definition (Kudo-Araki, Dyer-Lashof)

Let $f : M \to \overline{\mathrm{Conf}}_n(\mathbb{I}^\infty)$ represent a homology class x.
Then $q_i(x)$ is represented by a map from $S^i \times_{\mathcal{S}_2} (M \times M)$ to $\overline{\mathrm{Conf}}_{2n}(\mathbb{I}^\infty)$ sending (v, m_1, m_2) to $\frac{1}{3}(f(m) - v) \coprod \frac{1}{3}(g(n) + v)$.

Homology of symmetric groups

These are non-trivial even (especially) on the non-zero class ι in $H_0(B\mathcal{S}_1)$, in which case the $q_i(\iota)$ are the homology of $B\mathcal{S}_2 = \mathbb{R}P^\infty$.

One can of course compose these operations, with the following a picture of $q_1(q_2(\iota))$.



Homology of symmetric groups

These operations are close cousins to Steenrod operations.

$$\text{(Adem)} \;\; \text{For} \;\; m > n, \;\; q_m \circ q_n = \sum_i \binom{i-n-1}{2i-m-n} q_{m+2n-2i} \circ q_i.$$

Given a sequence $I = i_1, \cdots, i_k$ of non-negative integers, let $q_I = q_{i_1} \circ \cdots \circ q_{i_k}$. Using the Adem relations, we can reduce to q_I whose entries are non-decreasing. We call such an I *admissible*. If such an I has no zeros we call it *strongly admissible*.

Homology of symmetric groups

Theorem (Nakaoka; Kudo-Araki; Cohen-Lada-May)

$H_*(\coprod_n B\mathcal{S}_n)$, as a ring under $*$ is the polynomial algebra generated by the nonzero class $\iota \in H_0(B\mathcal{S}_1)$ and $q_I(\iota) \in H_*(B\mathcal{S}_{2^k})$ for I strongly admissible.

Homology and cohomology of symmetric groups

It is useful to compare this well-known polynomial ring basis with the Hopf monomial (skyline) basis for cohomology.

The pairing decomposes into blocks by shape, which are only nonzero on the "diagonal".

The basic pairings to compute are then the products of $\gamma_{i[2^j]}$ on the q_I, where $i + j$ equals the length of ℓ.

Homology and cohomology of symmetric groups

The cup coproduct $\Delta_\cdot$ on $H_*(\coprod_n B\mathcal{S}_n)$ is given by extending the formula for I admissible $\Delta_\cdot(q_I) = \sum_{J+K=I} q_J \otimes q_K$, where when $I = i_1, \cdots, i_n$ we have that J and K range over partitions of the same length such that for each ℓ, $j_\ell + k_\ell = i_\ell$.

This coproduct is more complicated than it seems at first. Even when starting with an admissible I, the sum above is over all possible J and K. Thus to get an expression in the standard basis, as needed for example to apply the coproduct again, one must apply Adem relations. The ones which get used most often are the relations $q_{2n+1}q_0 = 0$ and $q_{2n}q_0 = q_0 q_n$.

For example,

$$\Delta_\cdot q_{2,2} = q_{1,1} \otimes q_{1,1} + q_{2,0} \otimes q_{0,2} = q_{1,1} \otimes q_{1,1} + q_{0,1} \otimes q_{0,2},$$

(plus a symmetric term).

Outline of proofs

Definition

γ_ℓ is the linear dual of $q_{1,\ldots,1}$ in the Nakaoka (Kudo-Araki) monomial basis.

After setting up Hopf ring structure, first ingredient of proofs is the following. Let $\Delta_\odot$ be the coproduct dual to transfer product.

Theorem

$\Delta_\odot q_I = 0$

Corollary

The cohomology of symmetric groups is an exterior algebra under $\odot$, generated by $q_I^\vee$.

Outline of proofs

Theorem

$\Delta_{\odot} q_I = 0$

Proofs: Geometrically, transfer coproduct takes a family of configurations and sums over all ways to bicolor the configurations. For the q_I, any bi-coloring is at some level "asymmetric", in which case the "homology class over which one is taking a wreath product lifts to zero.

Alternately, the q_I are operations on divided powers, and the transfer coproduct correponds to the pinch map $D\underline{S^0} \to D\underline{S^0} \vee \underline{S^0}$, which can be analyzed directly.

Outline of proofs

The second ingredient is that the linear duals $q_I^\vee$ of fixed length k for I form a ring under $\cdot$ – the $\odot$-indecomposible ring on $B\mathcal{S}_{2^k}$ – which Cohen-Lada-May call R_k.

A simple combinatorial argument using the formula for Δ. shows that this ring is polynomial, generated by the linear duals $q_{0,\cdots,0,1,\cdots 1}$. Using coproduct, these are identified inductively with $\gamma_{i[2^j]}$.

That's basically it - once Hopf ring structure is identified, two ingredients, both of which date back forty or more years, are all that's needed!

Open question

The evaluation of the skyline basis on the Nakaoka basis is determined by that of the monomial basis for R_k on the q_I.
To our knowledge, this is not known!

$H^6/H_6(B\mathcal{S}_4)$	$q_{0,3}$	$q_{2,2}$
$\gamma_{1[2]}^3$	1	1
γ_2^2	0	1

Resolving this could be both useful for calculations, and could give either a new basis for the dual of the Steenrod algebra or shed new light on the Milnor basis.

See Open Questions for further discussion.

Summary

The brief Hopf ring description of the cohomology of symmetric groups readily leads to an additive basis and rules for multiplication.

Further structural understanding of the cup product structure is more complicated.

The proofs for the Hopf ring presentation mainly rely on long-known facts about the homology of symmetric groups - that is, the Kudo-Araki/ Dyer-Lashof algebra, as presented by Cohen-Lada-May.

The standard homology and cohomology bases pair non-trivially, in a way that is not currently understood.