Cohomology of symmetric groups

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Lecture notes, exercises, and open questions available at
https://pages.uoregon.edu/dps/VNUS2019/
which is linked to from my main web page.

Homology of symmetric groups

Theorem (Nakaoka; Kudo-Araki; Cohen-Lada-May)

$H_*(\coprod_n B\mathcal{S}_n)$, as a ring under $*$ is the polynomial algebra generated by the nonzero class $\iota \in H_0(B\mathcal{S}_1)$ and $q_I(\iota) \in H_*(B\mathcal{S}_{2^k})$ for I strongly admissible.

Homology and cohomology of symmetric groups

It is useful to compare this well-known polynomial ring basis with the Hopf monomial (skyline) basis for cohomology.

The pairing decomposes into blocks by shape, which are only nonzero on the "diagonal".

The basic pairings to compute are then the products of $\gamma_{i[2^j]}$ on the q_I, where $i + j$ equals the length of ℓ.

Homology and cohomology of symmetric groups

The cup coproduct $\Delta.$ on $H_*(\coprod_n B\mathcal{S}_n)$ is given by extending the formula for I admissible $\Delta.(q_I) = \sum_{J+K=I} q_J \otimes q_K$, where when $I = i_1, \cdots, i_n$ we have that J and K range over partitions of the same length such that for each ℓ, $j_\ell + k_\ell = i_\ell$.

This coproduct is more complicated than it seems at first. Even when starting with an admissible I, the sum above is over all possible J and K. Thus to get an expression in the standard basis, as needed for example to apply the coproduct again, one must apply Adem relations. The ones which get used most often are the relations $q_{2n+1}q_0 = 0$ and $q_{2n}q_0 = q_0q_n$.

For example,

$$\Delta. q_{2,2} = q_{1,1} \otimes q_{1,1} + q_{2,0} \otimes q_{0,2} = q_{1,1} \otimes q_{1,1} + q_{0,1} \otimes q_{0,2},$$

(plus a symmetric term).

Outline of proofs

Definition

γ_ℓ is the linear dual of $q_{1,\ldots,1}$ in the Nakaoka (Kudo-Araki) monomial basis.

After setting up Hopf ring structure, first ingredient of proofs is the following. Let $\Delta_\odot$ be the coproduct dual to transfer product.

Theorem

$\Delta_\odot q_I = 0$

Corollary

The cohomology of symmetric groups is an exterior algebra under $\odot$, generated by $q_I^\vee$.

Outline of proofs

Theorem

$\Delta_{\odot} q_I = 0$

Proofs: Geometrically, transfer coproduct takes a family of configurations and sums over all ways to bicolor the configurations. For the q_I, any bi-coloring is at some level "asymmetric", in which case the "homology class over which one is taking a wreath product lifts to zero."

Alternately, the q_I are operations on the divided powers functor, which extends to spectra. The transfer coproduct correponds to the pinch map $D\underline{S}^0 \to D\underline{S}^0 \vee \underline{S}^0$, which can be analyzed directly.

Outline of proofs

The second ingredient is that the linear duals $q_I^\vee$ of fixed length k for I form a ring under $\cdot$ – the $\odot$-indecomposible ring on $B\mathcal{S}_{2^k}$ – which Cohen-Lada-May call R_k.

A simple combinatorial argument using the formula for Δ. shows that this ring is polynomial, generated by the linear duals $q_{0,\cdots,0,1,\cdots 1}$. Using coproduct, these are identified inductively with $\gamma_{i[2^j]}$.

That's basically it - once Hopf ring structure is identified, these two ingredients (both of which date back forty or more years) are all that's needed!

Open question

The evaluation of the skyline basis on the Nakaoka basis is determined by that of the monomial basis for R_k on the q_I.
To our knowledge, this is not known!

$H^6/H_6(B\mathcal{S}_4)$	$q_{0,3}$	$q_{2,2}$
$\gamma^3_{1[2]}$	1	1
γ^2_2	0	1

Resolving this could be both useful for calculations, and could give either a new basis for the dual of the Steenrod algebra or shed new light on the Milnor basis. (We will indicate the connection in Lecture 4).

Restriction to subgroups, and Steenrod structure

The next stop on our journey is to understand Steenrod structure on $H^*(B\mathcal{S}_n)$, which by the Barratt-Priddy-Quillen-Segal theorem gives the Steenrod structure on $QS^0 = \Omega^\infty S^\infty$ – the space whose homotopy is the stable homotopy groups of spheres.

Our approach is to appeal to more "classical" (in two senses) approaches to group cohomology.

Restriction to subgroups, and Steenrod structure

For the moment, use the simplicial/ bar construction model for BG, a quotient of $\coprod G^n \times \Delta^n$.

We visualize $(g_1, \ldots, g_n) \times 0 \leq t_1 \leq \cdots \leq t_n \leq_1$ as as particles in $\mathbb{I}$, at the times t_i, labeled by elements of G so that the particle at t_i is labeled by g_i.

Informally, the identifications say that these particles fuse and their lables multiply when they collide, and disappear when they move to the end or are labeled by the identity element).

Restriction to subgroups

The simplicial BG is clearly functorial, compatibly with functorality for subgroups defined through quotient maps by using EG as a model for EH.

Moreover, in the simplicial model G acts on BG by conjugation – that is, having g conjugate each g_i.

Let $N(H)$ denote the normalizer of H in G, and let $W(H) = N(H)/H$.

Proposition

The restriction map $H^*(BG) \to H^*(BH)$ has image in the invariants of the latter under an action of $W(H)$.

Restriction to subgroups

Proposition

The restriction map $H^*(BG) \to H^*(BH)$ has image in the invariants of the latter under an action of $W(H)$.

The proof has two parts. A key lemma is that an action of G on simplicial BG by conjugation of all the labels is homotopically trivial – proven by an explicit homotopy to the identity. This same action of $W(H)$ on BH thus factors through an action of $W(H)/H$.

When the subgroup in question is elementary abelian, the restriction will be to $\mathbb{F}_2[x_1, \ldots, x_n]^{W(H)}$, an object of classical interest.

Quillen showed that restriction to all elementary abelian subgroups captures $H^*(BG)$ modulo nilpotent elements (under cup product).

Restriction to subgroups

The most important elementary abelian p-subgroups of symmetric groups are defined by considering the action of ${\mathbb{F}_p}^n$ on itself by addition. This defines an inclusion in $\mathcal{S}_{p^n}$ whose image we call V_n.

All elementary subgroups are images of $\prod V_{n_i}$ under the standard embedding of $\prod \mathcal{S}_{p^{n_i}}$ in $\mathcal{S}_m$.

The normalizer of V_n in $\mathcal{S}_{p^n}$ clearly contains a copy of conjugation by the n-dimensional affine group over $\mathbb{F}_p$, also embedded in $\mathcal{S}_{p^n}$ by action on ${\mathbb{F}_p}^n$ In fact $W(H) \cong GL_n(\mathbb{F}_p)$.

Thus the cohomology of BS_{2^n} restricts to the ring $\mathbb{F}_2[x_1, \ldots, x_n]^{GL_n(\mathbb{F}_p)}$.

Dickson algebras

Rings of invariants have long been of interest.
$\mathbb{F}_2[x_1, \ldots, x_n]^{GL_n(\mathbb{F}_p)}$ was first studied by Dickson (U Chicago) in this thesis over 100 years ago.

Theorem (Dickson)

$\mathbb{F}_2[x_1, \ldots, x_n]^{GL_n(\mathbb{F}_p)}$ is a polynomial algebra on classes $d_{k,\ell}$ in degree $2^k(2^\ell - 1)$ with $k + \ell = n$.

Let's call this algebra D_n.

The most "basic" generator is $d_{0,nl}$, which is the product of all linear combinations of the x_i (and thus obviously invariant).

Restriction to elementary abelian subgroups of symmetric groups

Theorem (Madsen-Milgram)

The direct sum of maps $res_{V_n} \oplus \Delta$ *from* $H^*(B\mathcal{S}_{2^n})$ *to* $D_n \oplus (H^*(B\mathcal{S}_{2^{n-1}}) \otimes H^*(B\mathcal{S}_{2^{n-1}}))$ *is injective.*

Corollary

Restriction from $H^*(B\mathcal{S}_m)$ *to all elementary abelian subgroups is injective.*

Therefore, there are no nilpotent elements in cohomology, as also follows from our explicit presentation.

Restriction to elementary abelian subgroups of symmetric groups

Theorem (Madsen-Milgram)

Under restriction, $\gamma_{i[2^j]}$ maps to the Dickson generator in that degree.

This gives another proof that the $\gamma_{i[2^j]}$ with $i + j = n$ generate a polynomial ring of $H^*(B\mathcal{S}_n)$.

In sum, restriction tells all and is computable!

Calculating the restriction to all elementary abelian subgroups leads to consideration of $(D_n^{\otimes k})^{\mathcal{S}_k}$, which in turn leads to the rings of invariants we discussed in the first lecture.

Hopf ring generators

	$B\mathcal{S}_2$	$B\mathcal{S}_4$	$B\mathcal{S}_6$	$B\mathcal{S}_8$	$B\mathcal{S}_{10}$	$B\mathcal{S}_{12}$	$B\mathcal{S}_{14}$	$B\mathcal{S}_{16}$
15								γ_4
14								$\gamma_{3[2]}$
13								
12								$\gamma_{2[4]}$
11								
10								
9						$\gamma_{2[3]}$		
8								$\gamma_{1[8]}$
7				γ_3			$\gamma_{1[7]}$	
6				$\gamma_{2[2]}$		$\gamma_{1[6]}$		
5					$\gamma_{1[5]}$			
4				$\gamma_{1[4]}$				
3		γ_2	$\gamma_{1[3]}$					
2		$\gamma_{1[2]}$						
1	γ_1							

Steenrod structure

Because transfers are stable maps, our transfer product has a Cartan formula - there are two!

Thus, Steenrod operations on all cohomology of symmetric groups is thus determined by that on the $\gamma_{i[j]}$.

By the injectivity of $res_{V_n} \oplus \Delta$, these in turn are determined inductively by the action of the Steenrod algebra on Dickson generators.

Steenrod structure

Hưng calculated the Steenrod squares on Dickson classes as given by

$$Sq^i d_{k,\ell} = \begin{cases} d_{k',\ell'} & i = 2^k - 2^{k'} \\ d_{k',\ell'} d_{k'',\ell''} & i = 2^n + 2^k - 2^{k'} - 2^{k''}, \quad k' \leq k < k'' \\ {d_{k,\ell}}^2 & i = 2^k(2^\ell - 1) \\ 0 & otherwise. \end{cases}$$

Steenrod structure

Definition

- The height of a skyline diagram is the largest number of blocks stacked in a column. (This is not the degree of that column.)
- The effective scale of a column is the index of the largest block-type which occurs. (The largest i for which $\gamma_{i,2^j}$ appears in the column.) The effective scale of a skyline diagram is the minimum of the effective scales of its columns.
- We say a skyline diagram is not full width if iit has an empty space. (That is, it represents a non-trivial transfer product of some monomial with some 1_k.)

Steenrod structure

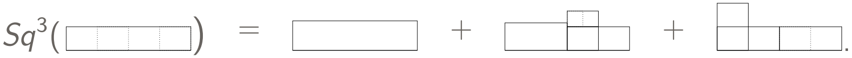
Theorem

$Sq^i \gamma_{\ell[2^k]}$ is the sum of all full-width skyline diagrams of total degree $2^k(2^\ell - 1) + i$, height one or two, and effective scale at least ℓ.

We call monomials represented by such skyline diagrams the outgrowth monomials of $\gamma_{\ell[2^k]}$.

For example,

$$Sq^3 \gamma_{2[4]} = \gamma_4 + \gamma_3 \odot \gamma_2 \gamma_{1[2]} \odot \gamma_2 + {\gamma_2}^2 \odot \gamma_2 \odot \gamma_{2[2]}.$$



Steenrod structure

Again, the proof is to use injectivity of $res_{V_n} \oplus \Delta$ for an induction, with Hu'ng's result as the base.

The first example (beyond $\mathbb{R}P^\infty$) is instructive.

To calculate $Sq^1\gamma_{1[2]}$ we first calculate that under $res_{V_2} \oplus \Delta$ it maps to $d_{1,1} \oplus (\gamma_1 \otimes \gamma_1)$.
$Sq^1 d_{1,1} = d_{2,0}$, the top Dickson class. While
$Sq^1(\gamma_1 \otimes \gamma_1) = \gamma_1^2 \otimes \gamma_1 + \gamma_1 \otimes \gamma_1^2$, by the Cartan formula.

The unique class which maps to $d_{2,0} \oplus (\gamma_1^2 \otimes \gamma_1 + \gamma_1 \otimes \gamma_1^2)$ under $res_{V_2} \oplus \Delta$ is $\gamma_2 + \gamma_1^2 \cdot \gamma_1$, so this is $Sq^1\gamma_{1[2]}$.

Steenrod structure

The Dickson algebras are naturally quotients of $H^*(B\mathcal{S}_{2^n})$. Those quotient maps can be split, formally.

This splitting does not preserve Steenrod operations – because of the terms which are $\odot$-decomposible in the formulae for Sq^i. Moreover, no splitting preserving Steenrod operations can exist.

We conjecture that while $\oplus E^*(B\mathcal{S}_n)$ is always a Hopf ring, only in cohomology will the rings of indecomposibles be split.

The transfer product filtration

We think that this Steenrod structure is ripe for further study. We make more precise the intimate connection with Dickson algebras.

By definition, there is an exact sequence

$$0 \to \odot - \text{Decomposibles} \to H^*(B\mathcal{S}_{2^n}) \to D_n \to 0.$$

For $n = 2$ this reads

$$0 \to \wedge^2 H^*(B\mathcal{S}_2) \to H^*(B\mathcal{S}_4) \to D_2 \to 0,$$

which is simple to interpret through skyline diagrams.

The transfer product filtration

A dyadic partition P of an even number m is an expression $m = \sum s_i 2^i$ where $i \geq 1$. Let $\mathrm{Par}(m)$ denote the set of such. In general, there is an additive decomposition

$$H^*(B\mathcal{S}_m) \cong \bigoplus_{P \in \mathrm{Par}(m)} (\bigotimes_i \wedge^{s_i} D_i).$$

The formulae above show that this is **as far from a direct sum decomposition over the Steenrod algebra as possible**.

Instead, we see the right-hand side as the associated graded of a filtration by number of transfer products, which has extensions at all depths.

The transfer product filtration

As we see below, the cohomology of $B\mathcal{S}_m$ is a split summand of that of $B\mathcal{S}_\infty$ as an $\mathcal{A}$-module, which by the Barratt-Priddy-Quillen-Segal theorem is the cohomology of $\Omega^\infty\Sigma^\infty S^0$.

Problem: understand the topological significance of and make calculations of the extensions of Dickson algebras which occur in the transfer product filtration.

Of course, to make calculations one must first study tensor and exterior powers of Dickson algebras themselves.