Cohomology of symmetric groups

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Cohomology of alternating groups

Much less was known about alternating groups before our work.

For example, because of injectivity of collective restriction to elementary abelian subgroups, it was known that there were no nilpotent elements in cohomology rings of symmetric groups. This property for alternating groups was not known before our work. (It's true.)

Building on our understanding of symmetric groups, we present the cohomology of alternating groups. But we require all techniques, some extended, simultaneously.

Cohomology of alternating groups

The four ingredients/ techniques we employ are

- the Gysin sequence, and our calculation for symmetric groups
- (almost) Hopf ring structure
- Fox-Neuwirth resolutions
- restriction to elementary abelian subgroups.

First ingredient - the Gysin sequence

$B\mathcal{A}_n$ is a double-cover of $B\mathcal{S}_n$ – in order words, a cover with S_0 fiber.

Recall that the Gysin sequence involves mulitplication by the Euler class. By the fact that this cover is pulled back from the $n = 2$ case through the sign homomorphism, the Euler class is $e = \gamma_1 \odot 1_{n-1}$.

The Gysin sequence in this case yields short exact sequences

$$0 \to H^*(B\mathcal{S}_n)/e \xrightarrow{\text{res}} H^*(B\mathcal{A}_n) \xrightarrow{\text{tr}} \mathrm{Ann}(e) \to 0,$$

where $\mathrm{Ann}(e)$ is the annihilator ideal of $e = \gamma_1 \odot 1_{n-1}$.

The Gysin sequence

Here we can use our skyline diagrams. $e = \gamma_1 \odot 1_{n-1}$ is a 1×1-block which can be stacked on any column of width one (including empty ones).

Its annihilator ideal consists of skyline diagrams which are "full" where each column has at least one element of a larger block type – that is $\gamma_{k[m]}$ for $k > 1$.

For a diagram with column of width one which is higher than all others, one can add the product of e with the diagram obtained by removing a block from that column. The resulting sum will have a reduced difference of height between highest and next highest columns. Inductively, the tallest column of 1×1 blocks must be repeated - that is occur as a $\gamma_{1[k]}^m$ with $k > 1$.

The Gysin sequence

In summary, the annihilator ideal has a basis of diagrams filled with columns of width two and filled by larger blocks.

The quotient has a basis of representatives whose tallest column of 1×1 blocks has width at least two.

There is a lot of overlap between these basis elements, which is well-understood in light of an additional piece of structure, namely, $B\mathcal{A}_n$ as a double-cover of $B\mathcal{S}_n$ has an involution on its cohomology. The image of restriction from $B\mathcal{S}_n$ is fixed under this involution.

The Gysin sequence, and the involution

Recall that exact sequences of $\mathbb{F}_2$-vector spaces with involution starting with a fixed space decompose as a direct sum of either

$$0 \to \langle x^0 \rangle \to \langle x^0 \rangle \to 0 \to 0$$

or

$$0 \to \langle x^+ + x^- \rangle \to \langle x^+, x^- \rangle \to \langle x^+ \sim x^- \rangle \to 0.$$

Here $\langle \cdot \rangle$ denotes span, and the involution fixes x^0 and switches x^+ to x^-.

So we will see two kinds of cohomology basis elements – *charged* which are mapped non-trivially under involution and *neutral* which are fixed. And our Gysin sequence includes the neutral classes (including the sum of a charged pair) into all classes, and then the quotient is the charged classes identified.

Second ingredient - Hopf ring structure

Definition

A product series of finite groups is a collection $\{G_i\}_{i \geq 0}$ with embeddings $e_{n,m} : G_n \times G_m \hookrightarrow G_{n+m}$ which are associative and commutative up to conjugation.

Theorem (Giusti-S)

The generalized cohomology of a product series of groups forms a (derived) almost Hopf ring. For alternating groups, the deviation from being a Hopf ring is controlled by a polarization operator.

Almost Hopf ring? What breaks down?

The proof is "functorial" – that is, established through commuting diagrams of spaces or spectra – for all of the properties which define an almost Hopf ring.

The one property not satisfied in general is that $(\odot, \Delta)$ form a bialgebra. In the case of symmetric groups, this does occur because

$$(\mathcal{S}_i \times \mathcal{S}_j)\backslash \mathcal{S}_n/(\mathcal{S}_k \times \mathcal{S}_\ell) = \bigsqcup \mathcal{S}_n/(\mathcal{S}_p \times \mathcal{S}_q \times \mathcal{S}_r \times \mathcal{S}_s).$$

For other series of groups there is no such equality, though alternating groups are not far.

Ultimately this is good news: Hopf ring distributivity is the real "work horse," and this failure just requires more input of coproduct information.

Hopf ring structures

Principle: natural ring-valued invariants of series of groups including the following tend to form almost Hopf rings.

- Group cohomology
- Representation rings
- Rings of invariants of standard families of representations
- Generalized cohomology (e.g. n-characters).

Hopf ring structure and involution

It is convenient to set $B\mathcal{A}'_0 = S^0$ so it double-covers $B\mathcal{S}_0$.

Then we incorporate the involution into Hopf ring structure by calling the generators of H^0 by 1^+ and 1^-, with the former being an identity for transfer product and transfer product with the latter defined to be involution.

The involution also allows us to define a polarization operator ρ which allows us to pick half of the terms in coproducts.

Main theorem

Theorem (Giusti-S)

$\bigoplus H^*(B\mathcal{A}_n)$ is the divided power component almost-Hopf ring under cup and transfer products generated by classes

$$\gamma_\ell^+ \in H^{2^\ell-1}(B\mathcal{A}_{2^\ell}) \;\; 2 \leq \ell,$$

$$\gamma_{1,k;m} \in H^k(B\mathcal{A}_{2m}) \;\; 2 \leq k \leq m \qquad \text{and } 1^- \in H^0(B\mathcal{A}'_0),$$

Relations between transfer products are

$$1^- \odot 1^- = 1^+ \tag{1}$$

$$1^- \odot \gamma_{1,k;m} = \gamma_{1,k;m} \tag{2}$$

$$\prod \gamma_{1,k;m} \odot \prod \gamma_{1,\ell;n} = 0, \tag{3}$$

Main theorem

Theorem (continued)

Let $\gamma^-_{\ell,m}$ *denote* $1^- \odot \gamma^+_{\ell,m}$*, and by convention set* $\gamma^\pm_{\ell,0} = 1^\pm$ *and* $\gamma_{1,1;m} = 0$*. Then cup relations are*

$$\gamma^+_{\ell[m]} \cdot \gamma^-_{k[n]} = 0 \textit{ unless } k = \ell = 2, \tag{4}$$

$$\gamma^+_{2[m]} \cdot \gamma^-_{2[m]} = \begin{cases} \left(\gamma^+_{2[m]} + \gamma^-_{2[m]}\right)^2 + \left(\gamma^+_{2[m-1]}\right)^2 \odot (\gamma_{1,2;2})^3 & (\mathrm{odd}) \\ \left(\gamma^+_{2[m-1]}\right)^2 \odot (\gamma_{1,2;2})^3 & m\ (\mathrm{even}) \end{cases} \tag{5}$$

$$\gamma^+_{\ell[m]} \cdot \gamma_{1,k;m2^{\ell-1}} = \begin{cases} (\gamma^+_{\ell[q]} \cdot \gamma_{1,k;k}) \odot \gamma_{\ell[m-q]} & \textit{if } k = 2^{\ell-1} \cdot q \\ 0 & \textit{otherwise.} \end{cases} \tag{6}$$

Main theorem

Theorem (continued)

While $\bigoplus H^*(B\mathcal{A}_n)$ does not form a Hopf ring, for any α and β there is in general the equality

$$\Delta(\alpha \odot \beta) = \Delta\alpha \odot_{\rho^+} \Delta\beta, \tag{7}$$

where $\odot_{\rho^+}$ is transfer product after applying the polarization operator.
Basic coproducts are

$$\Delta\gamma^+_{\ell[m]} = \sum_{i+j=m} \left(\gamma^+_{\ell[i]} \otimes \gamma^+_{\ell[j]} + \gamma^-_{\ell[i]} \otimes \gamma^-_{\ell[j]}\right) \tag{8}$$

$$\Delta\gamma_{1,k;m} = \sum \gamma_{1,p;i} \otimes \gamma_{1,q,j}, \tag{9}$$

where the last sum is over i, j, p, q with $i + j = m$ and $p + q = k$, where $0 \le p \le i$ and $0 \le q \le j$.

Main theorem - highlights

The Hopf ring generators for alternating and symmetric groups predominantly map to each other in the Gysin sequence, with level or "box" size preserved.

While the description of Hopf ring structure for symmetric groups is uniform, small levels are exceptional for alternating groups.

At level three or greater, there are two Hopf ring generators - a positive and a negative - for alternating groups for each generator for symmetric groups. These annihilate each other under cup product.

The charged classes at these level are "unstable", in that only the sum of such pairs lift to larger alternating groups (as 1_n is neutral, and neutral transfer charged is neutral).

Main theorem - highlights

At level two, there are still two sets of generators, but instead of annihilating each other under cup product there are exceptional relations which start at $\mathcal{A}_4$ and then propagate, as determined by coproduct structure.

Finally, at level one only one set of generators occurs, but we need a separate set of generators for each component. In comparison with symmetric groups that set lacks the degree-one generator, which is the Euler class in the Gysin sequence.

Transfer products are relatively simple, with relations mostly governed by a notion of charge which implies that the transfer product of neutral classes vanishes.

Cup products are complicated, with complexity driven by both basic relations and Hopf ring structure, in contrast to the setting of symmetric groups where the latter alone occurs.

Third ingredient - Fox-Neuwirth cochains

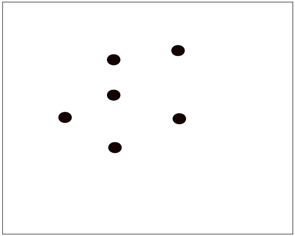
As one might surmise from the complexity of the relations, proofs are substantially more involved than they were for symmetric groups. Finer tools need to be developed.

Recall the $\mathrm{Conf}_n(\mathbb{R}^d)$ model for $E\mathcal{S}_n$, and thus $E\mathcal{A}_n$, through a range. We put a cell structure on it based on the dictionary ordering of points in $\mathbb{R}^m$ using standard coordinates, which we denote by $<$. This ordering gives rise to an ordering of points in a configuration.

Definition

Let the ith (depth of) agreement of a configuration be the number of consecutive coordinates shared by the ith and $i+1$st points (in the dictionary order) of a configuration.

Third ingredient - Fox-Neuwirth cochains



For example, this configuration in the plane has agreements $[0, 1, 1, 0, 1]$.

Fox-Neuwirth cochains

Theorem (after Fox-Neuwirth)

For any list of $n-1$ non-negative integers $\lambda = [a_1, \ldots, a_{n-1}]$, the subspace of $Conf_n(\mathbb{R}^m)$ of configurations with λ as its list of agreements is homeomorphic to a Euclidean ball of dimension $mn - \sum a_i$. These subspaces are the interiors of cells in a CW structure on the one-point compactification $\overline{Conf}_n(\mathbb{R}^m)^+$.

By Alexander duality, the cellular chain complex computes the cohomology of $\overline{\mathrm{Conf}}_n(\mathbb{R}^m)^+$, which agrees through degree m with that of $\overline{\mathrm{Conf}}_n(\mathbb{R}^\infty) = B\mathcal{S}_n$.

Fox-Neuwirth cochains

While the cells are immediate to enumerate the differential in the cellular chain complex, which is thus an alternative to the standard bar complex, as first given explicitly by Giusti-S, is complicated, involving signed counts of shuffles.

$$
\begin{aligned}
\delta([2,0,1,2]) &= -3[2,0,2,2] - 1[2,1,1,2] \\
\delta([1,0,2,2]) &= -1[1,1,2,2] + 1[1,2,2,1] + 2[2,0,2,2] - 1[2,2,1,1] \\
\delta([0,2,1,2]) &= -6[0,2,2,2] - 1[1,2,1,2] + 1[2,1,1,2] - 1[2,1,2,1] \\
\delta([0,1,2,2]) &= -4[0,2,2,2] - 1[1,2,2,1].
\end{aligned}
$$

We also give a version “with orientations” for alternating groups.

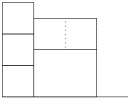
Fox-Neuwirth cochains

Originally due to Fox-Neuwirth in two dimensions (braid groups) this cochain complex and related category was first extended by Vassiliev and then revisited by Batanin, Joyal, and Ayala-Hepworth in the context of *n*-categories.

Giusti-S give explicit cocyle representatives for $H^*(B\mathcal{S}_n, \mathbb{F}_2)$ in the Fox-Neuwirth complexes.

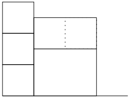
The γ_ℓ are represented by $[1, 1, \ldots, 1]$ and $\gamma_{\ell[m]}$ by $[1, 1, \cdots, 1, 0, 1, 1, \cdots 1, 0, \cdots]$.

Fox-Neuwirth cochains

Or consider $\alpha =$  .

It is represented by the mod-two cochain $\mathrm{BISym}[3, 0, 2, 1, 2, 0, 0]$, where BISym denotes the sum over all lists with the same set of consecutive collections of non-zero integers (so that for example $[0, 2, 1, 2, 0, 3, 0]$ would be another).

Digression: geometry of characteristic classes

$\alpha =$  defines a characteristic class in degree eight for an eight-sheeted covering map as follows for manifolds.

- Embed the covering $p : \widetilde{M} \to M$ in some $M \times \mathbb{R}^d \to M$.
- Consider the locus $\chi_\alpha(p) \subset M$ of m such that in $p^{-1}(m)$ there are two points which share their first three coordinates, and four points which share their first coordinate and can be partitioned into two groups of two which share their second coordinate.
- Then the characteristic class of p associated to α is Poincaré dual to $\chi_\alpha(p)$ (under standard transversality assumptions)

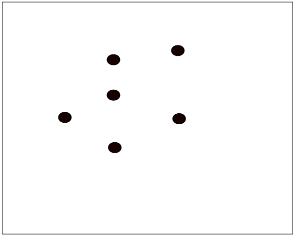
All characteristic classes of finite-sheeted covering spaces have similar descriptions in terms of agreement of coordinates in a fiber after embedding in a trivial Euclidean bundle.

Digression on Eilenberg-MacLane spaces

It is "well-known" that through a Pontrjagin-Thom construction $\overline{\mathrm{Conf}}_n(\mathbb{R}^d)$ maps to $\Omega^d S^d$, through a map which becomes a homology isomorphism in an appropriate limit. The Fox-Neuwirth cell structure is related through duality to the iterated cobar construction.

But Fox-Neuwirth cells (for configurations with labels if $A \neq \mathbb{Z}/2$) also naturally enumerate give a cell structure on the iterated bar model for Eilenberg-MacLane spaces.

Digression on Eilenberg-MacLane spaces

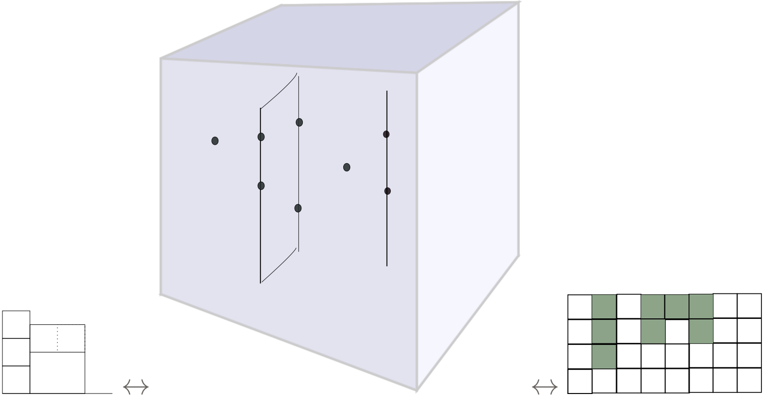


represents a point in $B^{(2)}\mathbb{Z}/2 = K(\mathbb{Z}/2, 2)$ labeled by $[g]|[g|g|g]|[g|g]$.

Proposition

There are isomorphisms of mod-two *cellular chain complexes of* $K(\mathbb{Z}/2, n)$ *and* $\bigvee_{n,d} \overline{Conf}_n(\mathbb{R}^d)$.

Digression on Eilenberg-MacLane spaces



The same collection of cells represents a cohomology class in $H^8(\overline{\mathrm{Conf}_8}(\mathbb{R}^4))$ and $H_{24}(K(\mathbb{Z}/2, 4))$ respectively.

Digression on Eilenberg-MacLane spaces

It is fun to undersand the isomorphism between our skyline basis and Cartan's calculation of $H_*(K(\mathbb{Z}/2, n); \mathbb{Z}/2)$, and thus have better chain-level (and geometric) understanding of the latter.

It might be fruitful to use these to better understand the dual of the Steenrod algebra at the chain level, or alternately to use (overlapping) work of Tamaki, Berger, Ayala-Hepworth, Blagojevich-Zeigler to obtain chain level understanding of Adem relations and higher cohomology operations (if that hasn't been done already).

Conjecture: under this isomorphism, γ_k is sent to ξ_k.

Back to Fox-Neuwirth cochains

For alternating groups, Fox-Neuwirth cochain representatives end up being much more complicated than they were for symmetric groups.

For example, which $[1, 1, 1]$ is a Fox-Neuwirth cocycle for symmetric groups $\mathcal{S}_4$, $[1, 1, 1]^+$ is not a cocycle for $\mathcal{A}_4$. Instead, γ_2^+ is represented by $[1, 1, 1]^+ + [2, 0, 1]^+ + [2, 0, 1]^-$.

These extra terms account for the involved relations for alternating groups. For γ_k the number of "correction terms" grows quadratically in k, and are of a form such as $[1, 1, 1, 0, 2, 0, 1, 1, 1, 0, 1, 0, 1, 1, 1]$. We find and make use of explicit such.

Fourth ingredient: a detection result restriction to elementary abelian subgroups

The irregularity of the $\gamma_k^{\pm}$ for small k reflects the irregularity of elementary abelian subgroups of $\mathcal{A}_{2^k}$.

For $k > 2$ there are two transitive elementary abelian subgroups up to conjugation, which we call $V_k^{\pm}$. For $k = 2$ there is only one. Of course $\mathcal{A}_{2^1}$ is trivial, so must be V_1.

Quang classified elementary abelian subgroups in general. One of our main results, which involved explicit evaluation of Fox-Neuwirth cochain representatives through geometry, is that a chosen subset of elementary abelian subgroups detects. This allows us to verify relations in the main result.

Example

Theorem

The mod-two cohomology of $\mathcal{A}_8$ is generated as a ring under cup product by the following classes.

Degree	2	3	4	5	6	7
Classes	σ_2	σ_3	σ_4		d_6^+	d_7^+
		d_3		$d_3 \odot \sigma_2$	d_6^-	d_7^-

Relations are:

- *Products of* σ_2, d_3, *or* $d_3 \odot \sigma_2$ *with* $d_7^\pm$ *are zero (six relations).*
- *Products of* σ_3 *with* d_3, $d_3 \odot \sigma_2$, $d_6^\pm$, *or* $d_7^\pm$ *are zero (six relations).*
- $d_6^+ \cdot d_7^- = d_6^- \cdot d_7^+ = 0$; $d_7^+ \cdot d_7^- = 0$.
- $(d_3 \odot \sigma_2)^2 = \sigma_2 \cdot d_3 \cdot (d_3 \odot \sigma_2) \ + \ {\sigma_2}^2 \cdot (d_6^+ + d_6^-) \ + \ (d_3)^2 \cdot \sigma_4$.
- $d_6^+ \cdot d_6^- = \sigma_2 \cdot d_3 \cdot (d_3 \odot \sigma_2) + (\sigma_2)^3 \cdot (d_6^+ + d_6^-) + \ d_3 \cdot \sigma_4 \cdot (d_3 \odot \sigma_2) + \sigma_2 \cdot \sigma_4 \cdot \left((d_3 \odot \sigma_2)^2 + d_6^+ + d_6^-\right)$.

Example

This is isomorphic to, but much smaller than, a computer generated presentation (and differs from an erroneous presentation in the literature).

We also give Steenrod structure in general and in this example.

We were not able to find multiplicative generators of all components. The column-by-column approach that one can take for symmetric groups cannot produce any charged classes, leaving a fun open question to ponder!