Cohomology of symmetric groups

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Odd primes

The cohomology of symmetric groups at odd primes goes through with the same techniques as for mod-two, with "amusing complications."

- Establish Hopf ring structure.
- Show that $\Delta_{\odot} q_I = 0$, which implies that $\bigoplus_n H^*(B\mathcal{S}_n; \mathbb{F}_p)$ is a divided powers algebra under $\odot$.
- Use the Cohen-Lada-May calcuation of the $\odot$-indecomposibes, and show they are split multiplicatively.

Odd primes

As elsewhere in topology, Bocksteins lead to non-uniformity.

At the prime two, γ_1 is represented in the configuration space model by "two points which share their first coordinate." The Bockstein on it is its square which is represented by "two points which share their first two coordinates."

At $p > 2$, because we need orientations, we consider our configurations in $\mathbb{C}^\infty$. In this case γ_1 is represented by "p points share their first complex coordinate", a condition which has codimension $2(p-1)$.

This is dual to a class called q_2. (In lower index notation q_i multiplies degree by p and then adds $i(p-1)$.)

Odd primes

The class γ_1 is the mod-p reduction of an integral class which is p-torsion. The cochain which cobounds its product with p is thus a mod-p cocycle, which is linear dual to the Bockstein on q_2, which is represented by the fundamental class of a Lens space.

We can see product structure inside the first ring of Mùi invariants

$$H^*(BC_p)^{W(C_p)} = \mathbb{F}_p[\beta^{-1}x, x]^{GL_1(\mathbb{F}_p)} \cong \mathbb{F}_p[\beta^{-1}x \cdot x^{p-2}, x^{p-1}].$$

Odd primes

More generally, the $\odot$-indecomposibles are more complicated, and the algebraic picture is not resolved by invariant theory, because the indecomposibles do not map isomorphically to the Mùi invariants.

Products of duals of multiple Bockstein-related classes depend only on "where" the Bocksteins occur.

Definition

Let S be a subset of $\{1, ..., k\}$ for some k. Let $|S|$ be the cardinality of S, and let $d_p(S) = 2p^k - \sum_{i \in S} 2p^{k-i} - 2 + |S|$.

Given $T \subseteq \{1, \ldots, k\}$ such that S and T are disjoint, let $\langle S, T \rangle$ be sign of the permutation which relates the ordering of $S \smile T$ with S first (in order) then T and the ordering of $S \smile T$ as a subset of $\{1, \ldots, k\}$.

Odd primes

Theorem (Guerra)

For $p > 2$, $\bigoplus H^*(\mathcal{S}_n; \mathbb{F}_p)$ is a divided powers component Hopf ring generated by classes $\gamma_{S,k} \in H^{d_p(S)}(B\mathcal{S}_{p^k})$, where S is a subset of cardinality zero, one or two of $\{1, \dots, k\}$. Relations are:

- $\gamma_{S,k[p^i]} = \gamma_{S,k+i}$ *for all S such that $|S| = 2$;*
- $\gamma_{S,k} \cdot \gamma_{T,k} = 0$ *whenever S and T intersect non-trivially;*
- $(-1)^{\langle S,T \rangle} \gamma_{S,k} \cdot \gamma_{T,k} = (-1)^{\langle S',T' \rangle} \gamma_{S',k} \gamma_{T',k}$, *whenever S and T are disjoint, as are S' and T', and $S \smile T = S' \smile T'$.*

Moreover, $\gamma_{S,k}$ is primitive if either $|S| < 2$ or if $|S| = 2$ and $k \in S$.

Odd primes

The classes $\gamma_{\varnothing,k}$ are completely analogous to the classes γ_k at the prime two, for example generating a sub-Hopf ring which is formally isomorphic. We at times abbreviate and use γ_k instead of $\gamma_{\varnothing,k}$.

The last relation implies that non-zero products $\prod \gamma_{S_i,k}$ up to sign depend only on the union of the S_i and the set, with multiplicity, of k_i.

This leads to an additive basis of "decorated skyline diagrams" with multiplication by stacking, but with coefficient zero when labels overlap.

Odd primes

The rings of $\odot$-indecomposibles were calculated by Cohen-Lada-May, but our presentation is new. Our S - like their explicit "ij" – records the places of the Bocksteins in the classes $q_0 \cdots q_0 \beta q_1 \cdots q_1 \beta q_2 \cdots q_2$, the linear dual of which (in the standard basis) is $\gamma_{S,k}$.

As at the prime two, computing the product on the dual of admissable sequences of this length quickly leads to the duals of such classes as generators, through repeated application of $\Delta . q_I = \sum q_J \otimes q_K$. The facts that $\beta^2 = 0$ and that one multiple Bocksteins can be decomposed in all possible ways gives the relations.

Divided powers

Next we consider $HDP^*(X;k) = \bigoplus_n H^*(ES_n \times_{\mathcal{S}_n} X^n; k)$.

In the configuration model, the points in the configuration are now labeled by points in X. The cycles are Kudo-Araki-Dyer-Lashof cycles whose labels are cycles in X. The homology is the free algebra generated by the homology of X.

The cocycles are Fox-Neuwirth cocycles labeled by cocycles in X, so we have.

Theorem (Guerra-Salvatore-S)

A basis for $HDP^*(X;\mathbb{F}_2)$ is given by pairs $D, x_1 \otimes \cdots \otimes x_m$ where D is a skyline diagram and x_i are cohomology classes in X, one for each column in D. The degree is $\deg D + \sum 2 \cdot \mathrm{width}_i \cdot \deg x_i$ where width_i is the width of the column in D to which x_i corresponds.

Divided powers

Mutliplicatively, the result is clean.

Theorem (Guerra-Salvatore-S)

The direct sum of cohomology of divided powers of X, namely $HDP^*(X; \mathbb{F}_2)$ is a free divided-powers component Hopf ring primitively generated by the pull-back of classes $\gamma_\ell \in H^{2^\ell - 1}(B\mathcal{S}_{2^\ell})$ and the cohomology of X.

Divided powers at odd primes

Odd primes are trickier! In homology, if x is in odd degree, then the operations on x are of the form $q_{I_1 \beta I_2}$ where I_1 consists of even indices and I_2 of odd. This is different from even degree where operations are of the form $q_{I_1 \beta I_2 \beta I_3}$, where I_1 and I_3 are even and I_2 is odd. (We saw these operations in homology of symmetric groups, applied to the degree 0 class ι)

This is best incorporated by computing cohomology of the sign representation (which occurs as in cellular homology of odd cells). We consider $\bigoplus H^*(B\mathcal{S}_n; \mathbb{F}_p \oplus \mathrm{sgn})$ as a ring over $\bigoplus H^*(B\mathcal{S}_n; \mathbb{F}_p)$

Definition

For S a subset of $\{1, ..., k\}$, let $e_p(S) = p^k - \sum_{i \in S} 2p^{k-i} - 1 + |S|$.

Divided powers at odd primes

Theorem (Guerra-Salvatore-S)

The cohomology of $\mathbb{F}_p \oplus \mathrm{sgn}$ *over all symmetric groups is generated over that of* $\mathbb{F}_p$ *as a divided powers component Hopf ring by classes* $\gamma'_{S,k} \in H^{e_p(S)}(\mathcal{S}_{p^k}; \mathrm{sgn})$*, with* $|S| \leq 1$*, where the non-trivial divided powers of* $\gamma'_{\varnothing,k}$ *are all* 0*.*
A complete set of relations is given as an algebra over trivial coefficients:

- $\gamma'_{S,k[p^i]} = \gamma'_{S,k+i}$ *for all* $S \subseteq \{1, \ldots, k\}$ *such that* $|S| = 1$
- $\gamma'_{S,k} \cdot \gamma'_{T,k} = 0$ *and* $\gamma_{S,k} \cdot \gamma'_{T,k} = 0$ *whenever* $S \cap T \neq \varnothing$*.*
- $\gamma'_{S,k} \cdot \gamma'_{T,k} = (-1)^{\langle S,T \rangle} \gamma_{S,k} \cdot \gamma_{T,k}$*.*
- $\gamma_{S,k} \cdot \gamma'_{T,k} = (-1)^{\langle S,T \rangle + \langle S',T' \rangle} \gamma_{S',k} \cdot \gamma'_{T'_k}$ *whenever* S *and* T *are disjoint, as are* S' *and* T'*, and* $S \smile T = S' \smile T'$*.*

Moreover, $\gamma'_{\varnothing,k}$ *and* $\gamma'_{\{k\},k}$ *are primitive.*

Divided powers at odd primes

We can represent Hopf ring monomials as diagrams, again. Now, columns can be labeled as even or odd.

Theorem (Guerra-Salvatore-S)

A basis for $HDP^*(X; \mathbb{F}_p)$ is given by pairs $D, x_1 \otimes \cdots \otimes x_m$ where D is a Skyline diagram for $\mathbb{F}_p \oplus \mathrm{sgn}$ and x_i are cohomology classes in X, one for each column in D, whose parity agrees with the column. The degree is $\deg D + \sum 2 \cdot \mathrm{width}_i \cdot \deg x_i$ where width_i is the width of the column in D to which x_i corresponds.

The algorithm for multiplication is as you would think. This is work in progress, and we do not yet have Hopf ring generators and relations. We conjecture that the best description would be as a sub-object of the cohomology with coefficients in $\mathbb{F}_p \oplus \mathrm{sgn}$.

$\mathcal{S}_\infty$

An inverse limit over restriction maps, which stabilizes, gives the cohomology of $B\mathcal{S}_\infty$.

Restriction sends a diagram with enough space – that is, a Hopf ring monomial with an $\odot 1_r$ with r as large as the difference in size of symmetric groups – to "the same diagram with space removed" – that is, 1_r is replaced by $1_{r-(m-n)}$.

Diagrams without enough space are sent to zero.

S_∞

This is split by “adding space”. Because of the two Cartan formulae, this splitting respects $\mathcal{A}$-module structure. This splitting is not multiplicative, as we elaborate below.

The inverse limit can be represented by diagrams with infinite (or no additional) space. Algebraically we call these $m \odot 1_\infty$ where m is a Hopf ring monomial (with no $\odot 1_r$'s).

While this is not a Hopf ring, the graphical “stacking” algorithm still holds. But notice that there's now always room.

S_∞

Reflecting, “lack of room” causes all relations for cohomology of finite symmetric groups. Hu’ng, Adem-McGinnis-Milgram and Feshbach all noticed that relations vanish as one lifts classes.

For example, $\gamma_2 \cdot (\gamma_1 \odot 1_2) = 0$ because the (reduced) coproduct of γ_2 is trivial. But $(\gamma_2 \odot 1_2) \cdot (\gamma_1 \odot 1_4) = \gamma_2 \odot \gamma_1$.

In the limit, there are no relations – as first proven by Nakaoka – because there’s always room!

$\mathcal{S}_\infty$

A skyline diagram for $\mathcal{S}_\infty$ is the cup product of its columns, plus (many) terms of lower width.

So unlike for finite symmetric groups, where for example $\gamma_2 \odot \gamma_{1[2]}$ is indecomposible in $H^*(B\mathcal{S}_8)$, ring generators need only have one column.

When a skyline diagram is squared, the columns double (because any stacking of columns on the side will result in repeated columns). Thus, a column with even numbers of each block type will be a square.

Definition

A level-n Dickson partition of p is an equality $p = \sum_{k+\ell=n} t_k \left(2^k(2^\ell - 1)\right)$, where at least one of the positive integers t_k is odd. Consistent with Feshbach, we denote such a partition $\Lambda(n; \mathbf{t})$ or just Λ.
To such a Λ let m_Λ be the product $\prod \gamma_{\ell[k]}^{t_k}$.

Theorem

$H^*(B\mathcal{S}_\infty)$ is a polynomial ring, generated by the $m_\Lambda \odot 1_\infty$.

But now we see multiple filtrations - both the usual multiplicative depth, which is closely tied to width, and which γ_k occur, and their interrelationship.

As discussed above in the finite case, typically tensor products and exterior products of Dickson algebras occur as associated graded, leading to renewed interest in their structure.

QX

Theorem (after Dũng for $p = 2$)

Let X be a connected space of finite type. Then $H^*(QX; \mathbb{F}_p)$ is the graded commutative $\mathbb{F}_p$-algebra generated by classes of the form $b \odot 1^{[*]}$, where b is a decorated one-column diagram satisfying at least one of the following:

- *the decoration of the column is not a p^{th} power in $H^*(X; \mathbb{F}_p)$*
- *at least one of the constituent rectangles of the column appears a number of times that is not divisible by p*
- *the column has a non-empty S-decoration (if $p > 2$)*

Relations are $(b \odot 1^{[*]})^{p^{h(b)}} = 0$, with b even-dimensional. If x denotes the decoration of b, we define $h(b)$ through the following formula:

$$h(b) = \begin{cases} \min\{n \in \mathbb{N} : x^{p^n} = 0\} & \text{if } p = 2 \text{ or } b \text{ has no } S\text{-decoration} \\ 1 & \text{otherwise} \end{cases}$$

QX

Description of Steenrod structure is in progress. This is the flavor of the conjectured result.

Theorem

Let $x \in H^*(X; \mathbb{F}_p)$ Then

$$\mathcal{P}^i(x^{[k]}) = \sum \bigodot (\mathcal{P}^I x)^{[n_I]}.$$

We suspect that Hưng may have seen a related calculation.