Cohomology of symmetric groups

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Notes and other materials at:
https://pages.uoregon.edu/dps/VNUS2019/
Cảm ơn bạn!

Towards Morava K-theory of symmetric groups

Recall the Milnor ξ_i (our γ_i?) . It's linear dual plays important roles in homotopy - and is sometimes called P_i^0 and sometimes Q_i.

Explicitly, $Q_0 = Sq^1$ and $Q_i = [Q_{i-1}, Sq^{2^i}]$.

$(Q^i)^2 = 0$, meaning that $\mathrm{Ext}_{\Lambda Q^i}$ can see some non-nilpotent phenomena.

Towards Morava K-theory of symmetric groups

In topology, there are spectra of central importance - Brown-Peterson BP and Morava $K(n)$ with

$$H^*(BP) \cong \mathcal{A}//\Lambda(Q_0, Q_1, \cdots).$$

$$H^*(K(n)) \cong \mathcal{A}//\Lambda(Q_n).$$

The spectrum BP detects Nilpotence, and the $K(n)$ constitute the "primes" in stable homotopy. At a practical level, they are the only theories along with $H\mathbb{F}$ to have a perfect Künneth formua.

Towards Morava K-theory of symmetric groups

Margolis identified homology with respect to Q_i (and their cousins) as a key tool in the study of modules over the Steenrod algebra – for example, proving a "Whitehead Theorem" based on them.

In topology, Q_i-homology serves as an E_2 for an Adams or Atiyah-Hirzebruch spectral sequence for $K(n)^*(X)$.

Towards Morava K-theory of symmetric groups

Knowing $K(n)^*(BG)$ could provide further insight into $K(n)$ itself. Using restriction to elementary abelian subgroups, Hopkins-Kuhn-Ravenel identified the Euler characteristic in terms of n-characters. (Conjugacy classes of sets of n elements which commute with each other.) It is an open question as to how this is distributed.

Neil Strickland identified the Hopf ring indecomposibles of $\bigoplus_m E(n)^*(B\mathcal{S}_m)$ in algebro-geometric terms.

We are seeing that combining these techniques with our knowledge of ordinary cohomology and the Adams and/or(?) Atiyah-Hirzebruch spectral sequence can give further insight.

Q_i-homology of (cohomology of) $B\mathcal{S}_n$

Start with $B\mathcal{S}_4$ again. The standard decompsible/indecompsible exact sequence reads

$$0 \to \Lambda^2 H^*(B\mathcal{S}_2) \to B\mathcal{S}_4 \to D_2.$$

So we can compute Q_i-homology of both ends and calculate in a simple long exact sequence. We give an example, starting with the D_2 piece.
(Question: has anyone followed up on the significance on Q_i being iterated Frobenius (on $\mathbb{R}P^\infty$)?)

Q_i-homology of (cohomology of) D_n

The action of the Margolis operations on Dickson algebras is understood in general, and is often sparse or highly patterned. There are three quite distinct ranges of behavior for considering Q_m acting on the generators of D_n.

If $0 \leq m \leq n-2$, which we shall call the stable range, then we have

$$Q_m d_{2^k(2^l-1)} = \begin{cases} d_{2^n-1} & \text{for } k = m+1 \\ 0 & \text{otherwise.} \end{cases}$$

Q_i-homology of (cohomology of) D_n

If $m = n - 1$, which we call the critical case, then

$$Q_{n-1} d_{2^k(2^l-1)} = d_{2^n-1} \cdot d_{2^k(2^l-1)} \text{ for all } k.$$

If $m \geq n$, then

$$Q_m d_\bullet = Sq^{2^m} Q_{m-1} d_\bullet \text{ for inductive computation.}$$

Q_i-homology of (cohomology of) D_n

Since the Margolis operations are both differentials and derivations, it is obvious what the Margolis homology algebras are in the stable and critical cases

Theorem

1. *For* $0 \leq m \leq n-2$,

$$H_*(D_n; Q_m) = \mathbb{F}_2 \left[d_{2^{n-1}}, \ldots, d^2_{2^{m+1}(2^{n-m-1}-1)}, \ldots, d_{2^n-2} \right].$$

2. *For* $m = n-1$, $H_*(D_n; Q_{n-1})$ *is isomorphic to the (generally nonpolynomial) subalgebra of* D_n *with* $\mathbb{F}_2$*-basis consisting of all monomials in the generators* $d_{2^{n-1}}, \ldots, d_{2^n-2}$ *whose total exponent is even.*

Q_i-homology of (cohomology of) $B\mathcal{S}_n$

Preliminarily, we think we can show that $K(2)^*(B\mathcal{S}_4)$ is non-trivially distributed.
The biggest missing piece in the general picture is the following.

Conjecture

The Q_i homology of D_n when $i \geq n$ is $\mathbb{F}_2[d_{k,\ell}^2]/\left(Q_i(d_{2^n-1}), Q_i(d_{2^k(2^\ell-1)} \cdot d_{2^n-1})\right)$, as the $d_{2^k(2^\ell-1)}$ ranges over Dickson generators.

Q_i-homology of (cohomology of) $B\mathcal{S}_n$

The Margolis homology of exterior powers is relatively simple to analyze.

Definition

Let (V, d) be a chain complex, and consider exterior powers with the differential extended by the Leibniz rule. A d-double in $(\Lambda^2 V, d)$ is an element of the form $x \wedge dx$, obviously a d-cycle.

Proposition

Let (V, d) be a chain complex. The homology of the exterior algebra $(\Lambda^\bullet V, d)$ is generated as an algebra by the homology of V and the d-doubles.

Q_i-homology of (cohomology of) $B\mathcal{S}_n$

Observations:

- The $\odot$- decomposibles of $H^*(B\mathcal{S}_{2^n})$ are almost $\Lambda^2 H^*(B\mathcal{S}_{2^{n-1}})$. The distinction is not different conceptually.
- Margolis homology of Dickson algebras is simple when the i in Q_i is smaller than n in D_n and conjecturally simple in the other cases.
- Margolis homology of the cohomology of symmetric groups is a Hopf ring.
- There are differentials in the long exact sequence, but they seem entirely predictable.

So the conjecture on Margolis homology when the Q_i is large with respect to D_n (unstable) is the main step, with some "bookkeeping" all that's needed after that.

We would love to work with a Dickson algebra expert, perhaps along with a chromatic expert to finish off this project (these projects)!