Exercises and Investigations: Set 4

The "Exercises and Investigations" sets for this class are designed both to reinforce mathematical concepts and to lead you to think creatively about problems. You should clearly explain what you tried and how approached each item, even if you do not get to a final solution. Also, it often happens that you gain new insight into an old problem as time goes on and you are thinking about things from a new angle. So, as weeks go on, you may choose to go back and re-explore old problems in place of new ones.

1. Fix a prime number p. This problem concerns basic properties of the p-adic absolute value introduced in class. (The mathematical term for each property is given in parentheses.)

   (a) Show that $|a|_p \geq 0$ for all integers a, and $|a|_p = 0$ if and only if $a = 0$. (*nonnegativity*)

   (b) Show that $|ab|_p = |a|_p|b|_p$ for all integers a and b. (*multiplicativity*)

   (c) Show that $|a - b|_p \leq |a|_p + |b|_p$ for all integers a and b. (*triangle inequality)*

   Cool fact 1: A function that satisfies these three properties is called an *absolute value*. It turns out that you've now seen all possible absolute values. More precisely, the only absolute values are the usual absolute value, the p-adic absolute values, the *trivial absolute value* (the function that is 1 on all integers and 0 at 0), and powers of one of these absolute values! This fact is called Ostrowski's Theorem.

   Cool fact 2: It turns out that there is a whole field of p-adic geometry based on p-adic distances (replacing the usual absolute value by the p-adic absolute value). The p-adic absolute value and p-adic geometry turn out to play an important role in number theory. In fact, they play a role in the proof that for all positive whole numbers ≥ 3, there are no positive whole numbers a, b, c such that $a^n + b^n = c^n$.

2. What would go wrong in the above properties if we replaced p by a composite number in the p-adic absolute value? Why can't we use, say, 6 instead of a prime number p?

3. The p-adic absolute value obeys a particularly strong inequality, which is much stronger than the triangle inequality. Show that

   $$|a - b|_p \leq \max\left(|a|_p, |b|_p\right).$$

   [Hint: First, factor a power of p out of a and b. If you're still not sure what to do, try some examples with small numbers to try to gather some intuition for what is going on.]

4. The p-adic absolute value is useful in number theory because it measures divisibility by the prime p. Explain in what sense the p-adic distance $|a - b|_p$ between two integers a and b measures congruence mod powers of p.

5. Use the above properties to explain how you could extend the definition of the p-adic absolute value to the rational numbers. What should the value of $|1/2|_2$ be? What about $|44/21|_2$? $|21/44|_2$?

6. Try drawing the set of numbers 0 through 7 in a way that reflects their 2-adic distances from each other. Can you do this more generally? [Note: This is challenging. At least describe the approaches tried and what you found challenging, even if you don't come up with a completely satisfactory way to visualize this.]

7. Normally, we write our numbers in base 10. For example, we write 2463 to mean $2 \times 10^3 + 4 \times 10^2 + 6 \times 10 + 3 \times 10^0$. More generally, in base 10, $a_n a_{n-1} \cdots a_1 a_0$ denotes $a_n \times 10^n + \cdots + a_1 \times 10 + a_0 \times 10^0$. Instead, though, we could pick some other positive integer N and write our number in base N. In base N, the number $a_n a_{n-1} \cdots a_1 a_0$ denotes $a_n \times N^n + \cdots + a_1 \times N + a_0 \times N^0$. If we want to keep track of the p-adic absolute value of integers, it is convenient to work in base p. Explain why. If a number is given by $a_n a_{n-1} \cdots a_1 a_0$ in base p, what is its p-adic absolute value?

8. Bonus (Optional, not required): Write an exercise that could introduce someone to an aspect of the topic on which your collaborative project focuses. (This is an open-ended question that will likely require some thought.)