Exercises and Investigations: Set 1

The "Exercises and Investigations" sets for this class are designed both to reinforce mathematical concepts and to lead you to think creatively about problems. You should clearly explain what you tried and how approached each item, even if you do not get to a final solution. Also, it often happens that you gain new insight into an old problem as time goes on and you are thinking about things from a new angle. So, as weeks go on, you may choose to go back and re-explore old problems in place of new ones.

1. Number theorists are particularly interested in integers and fractions (and, more generally, to solutions to polynomial equations). To work in number theory, though, it often turns out to be useful to work with the complex numbers. In class, we introduced the complex plane, which is a fruitful place for exploratory analysis and discovery. This sequence of problems is designed to help you gain some intuition and also get more comfortable making conjectures and communicating about them.

   (a) What do you expect to happen in the complex plots of z, z^2, z^3, etc? Why? Now graph them. Do they conform to your expectations? Why do they behave this way? What about z^{-n} for n a positive integer?

   (b) In Figure 2.1 on p. 12 of *Visual Complex Functions* (from our online reserve reading list), there is a graph of function. Graph the function from the caption, and observe that your graph does not look like the one in the book. Why not? Figure out for which function Figure 2.1 is actually the graph.

   (c) Experiment with Mathematica's Manipulate function (or the equivalent, if you are using a different language) for various families of functions. Share (at least) two families you found particularly interesting. What did you observe in the behavior? What was expected? Unexpected? Do you understand why the families behave this way as you vary your parameter?

   (d) Go to the "Complex Function Plots" section and click on "Source" at `http://www.stevejtrettel.site/main/code.html`, and try to recreate Figure 1 from the next page. (Don't worry about the resolution or size of the picture you output.) How many of the features of the picture can you link to properties of the function? Are there phenomena (such as certain symmetries) that you observe but cannot explain? List those as well.

2. Prime numbers and factorization play a particular prominent role in number theory. Show that there are infinitely many prime numbers. [Hint: Let $p_1, \ldots, p_n$ be the first n prime numbers. What do you know about the prime divisors of $p_1 \cdots p_n + 1$?]

3. We are also interested in divisibility properties, and *modular arithmetic* (which we will encounter soon in class) provides a convenient approach to studying some divisibility questions. To get your mind in the right space for thinking about those question:

   (a) Show that for all odd n, 8 divides $n^2 - 1$.

   (b) Show that for every integer n, 30 divides $n^5 - n$ and 42 divides $n^7 - n$.

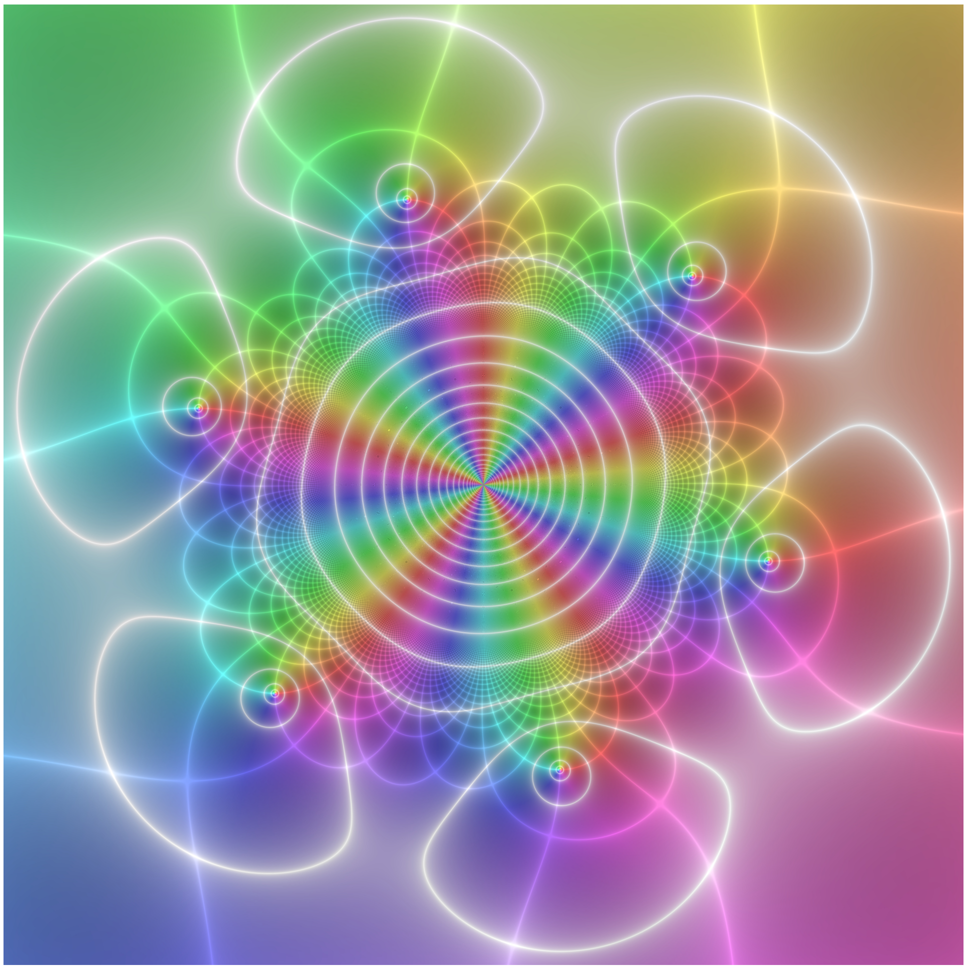

Figure 1: $f(z) = i(1/z)^5 + 2z$

4. The distribution of prime numbers is quite mysterious, yet there are certain predictable properties. Try to show that each of the following statements is true:

   (a) The square of every prime number $p \geq 5$ is exactly one more than a multiple of 24.

   (b) There are infinitely many numbers that are one more than a multiple of 24 and are not prime.

   (c) "Most" numbers that are one more than a multiple of 24 are not prime.

   (d) Let a and n be nonnegative integers. If $a^n - 1$ is prime, show that $a = 2$ and n is a prime. Such primes are called Mersenne primes. It's still unknown whether there are infinitely many Mersenne primes.

5. Is there a *nonconstant* polynomial f with integral coefficients such that $f(n)$ is prime for each integer n? Show that the answer is no.

6. We denote by $\pi(x)$ the number of prime numbers less than or equal to x. It turns out that $\pi(x) \equiv \frac{x}{\log x}$. Graph $\frac{x}{\log x}$. How does it compare to a polynomial function? (Conclude that a polynomial function would give a terrible estimate of $\pi(x)$.)