# Exercises and Investigations: Set 2

The "Exercises and Investigations" sets for this class are designed both to reinforce mathematical concepts and to lead you to think creatively about problems. You should clearly explain what you tried and how approached each item, even if you do not get to a final solution. Also, it often happens that you gain new insight into an old problem as time goes on and you are thinking about things from a new angle. So, as weeks go on, you may choose to go back and re-explore old problems in place of new ones.

1. In class, we showed that $\sqrt{2}$ is irrational. Adapt our strategy from class to show that if $p$ is a prime number, then $\sqrt{p}$ is irrational. Can you generalize this approach to show anything else is irrational? If so, what? (Note: This last question has intentionally been left vague to let you try to formulate a correct generalization, rather than just telling you what is or is not true.)

2. We've discussed several kinds of numbers in class, including complex, real, algebraic, and rational. Each of these sets of numbers is infinite, but it turns out that some of them are much larger than others, in the following sense. We say that an infinite set is *countable* (or *countably infinite*) if it has a one-to-one correspondence with the numbers $\{1, 2, 3, \dots\}$. In other words, we can count the elements of a countable set. In the following exercises, you may use the following facts (even if you don't justify them):

   - The union of a finite number of countable sets is countable.
   - The union of countably many countable sets is countable.
   - Every subset of a countable set is countable.
   - If sets $A$ and $B$ are countable, then the set of all pairs $(a, b)$ with $a$ an element of $A$ and $b$ an element of $B$ is countable.

   Show that

   (a) The set of rational numbers is countable. [Suggested strategy: Consider the array

   $$\begin{array}{cccc} 1/1 & 1/2 & 1/3 & \cdots \\ 2/1 & 2/2 & 2/3 & \cdots \\ 3/1 & 3/2 & 3/3 & \cdots \\ \vdots & \vdots & \vdots & \end{array}$$

   and enumerate the elements $1/1, 2/1, 1/2, 3/1, 2/2, 1/3, 4/1, 3/2, 2/3, \dots$, crossing off diagonals along the array...]

   (b) The set of real numbers is uncountable. [Suggested strategy: Suppose this set is countable. Then every subset is countable, so list all the positive real numbers $x_1, x_2, x_3, \dots$ between 0 and 1, and write them in decimal expansions (e.g. 0.12314....). Obtain a contradiction by finding a positive real number between 0 and 1 that can't be in your list.]

   (c) The set of irrational numbers is uncountable.

(d) The set of algebraic numbers of countable. [Suggested strategy: First, show that the set of polynomials with rational coefficients is countable. Then use the fact that a polynomial of degree $n$ with complex coefficients has at most $n$ zeroes in the complex numbers.]

(e) The set of transcendental numbers (i.e. the numbers that are not algebraic) is uncountable.

3. Show that $\log_4 21$ is irrational. Now, generalize your approach to construct infinitely many irrational numbers. How did you figure out how to generalize your approach? [Hint: Suppose $\log_4 21$ is rational, and obtain a contradition.]

4. In class, we stated that if $p(x)$ is a polynomial with complex coefficents (that is, $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$ with $a_0, \ldots, a_n$ complex numbers), then there is at least one complex number $\alpha$ such that $p(\alpha) = 0$. (This is called the Fundamental Theorem of Algebra.) In the Mathematica notebook GaussFTA.nb available on Canvas, there is a visual demonstration of the idea of a proof of this theorem, due to Gauss (from back in 1799!). Gauss claimed that if we look at the values of $p(x)$ on a circle of radius $r$, as $r$ grows from 0 to $\infty$, we can see that $p(re^{i\theta})$ must be zero for some $r$ and $\theta$. Try to explain in words why Gauss's argument works, and also try to explain how the demo in the Mathematica notebook helps illustrate it.

5. In "Exercises and Investigations" set 1, you showed that there are infinitely many prime numbers (using a strategy due to Euclid from around 300 BC). Now, refine the strategy from that exercise to show that there are infinitely many prime numbers congruent to $3 \mod 4$. Can you also extend your approach to another class of primes (such as primes congruent to a particular number mod some other integer)? If so, which one(s), and how? [Suggested starting point: Suppose there are only finitely many primes $p_1, \ldots, p_N$ congruent to $3 \mod 4$, consider the number $4p_2 \cdots p_N + 3$ (for example, $4 \cdot 3 \cdot 7 \cdot 11 \cdot 19 \cdots p_N + 3$), and show that this number has a prime factor congruent to $3 \mod 4$ that is bigger than $p_N$.]

6. BONUS problem (not required): In class, we used the fact that $e^{i\theta} = \cos\theta + i \sin\theta$. If you have had calculus, use the Taylor expansions of $e^x$, $\cos x$, and $\sin x$ to show that this formula makes sense for all real numbers $\theta$.