Research Statement

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1. Introduction

I work in number theory, and what interests me most are the connections and interactions between different areas of mathematics. My research is focused on automorphic forms, a central object in modern number theory with a broad range of applications. Notable examples include playing a vital role in the proof Fermat's Last Theorem, and in the Langlands program as a link between algebraic number theory and representation theory.

One may view automorphic forms as a class of holomorphic functions with special symmetries, or as representations of algebraic groups, or, from a geometric viewpoint, as sections of certain vector bundles. These varied but equivalent approaches suggest both the power of automorphic forms in relating these different areas of mathematics, and the wide range of techniques utilised in their study. For the work described below for example, my proof heavily involves aspects of harmonic analysis, and algebra and representation theory.

My research so far has been concentrated on a type of differential operator on automorphic forms (viewed as holomorphic functions). These operators are one of many tools used in the study of automorphic forms. However, one point of note is that they have found a wide variety of applications, ranging from proving certain algebraicity results to the study of mod-p representation theory. In §2 I discuss my recent results constructing analogous operators in a new setting, and in §3 I give some possible future directions for my research.

2. Recent Results on Rankin–Cohen Operators

In order to better contextualize my own work, I shall begin by briefly discussing the original Rankin–Cohen operators before moving on to the particular setting of my recent paper.

2.1. **In the case of modular forms.** The "base case" of an automorphic form is a *modular form*, which, from an analytic viewpoint, is a holomorphic function f on the complex upper half-plane $\mathbb{H}$ satisfying a boundedness condition and with a particular compatibility with the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ via Möbius transforms. More precisely, for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathbb{H}$ a modular form f needs to satisfy

$$f(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right),$$

where k is a fixed non-negative integer called the *weight* of f.

The derivative of a holomorphic modular form of integral weight on the complex upper half-plane is not in general a modular form since the derivative fails to satisfy the correct transformation properties. However, in 1956 R. A. Rankin was able to describe differential operators sending modular forms to modular forms [Ran56]. H. Cohen exhibited a special case in [Coh75], introducing particular bilinear differential operators on modular forms.

For two modular forms f and g of weights k and ℓ respectively and a fixed integer $v \geq 0$, the v-th *Rankin–Cohen bracket* or *Rankin–Cohen operator* of f and g is given by

$$[f, g]_v(z) := \sum_{r=0}^{v} (-1)^r \binom{v+k-1}{v-r} \binom{v+\ell-1}{v} f^{(r)}(z) g^{(v-r)}(z),$$

where, for the sake of convenience, we often take $f^{(r)}(z)$ to be $\frac{1}{(2\pi i)^r} \frac{d^r}{dz^r} f(z)$, a re-normalized r-th derivative of f. This particular normalization is chosen to preserve the coefficient ring of the modular forms. In particular, the Fourier coefficients of $[f, g]_v$ lie in any ring containing the Fourier coefficients of both f and g.

These Rankin–Cohen operators have several important properties, and have proven to be interesting objects to study. For instance, these operators provide the unique way of combining derivatives of two modular forms to produce another modular form of a higher weight. Specifically, for modular forms f and g of weights k and ℓ respectively, for each integer $v > 0$ the Rankin–Cohen bracket indexed by k, ℓ and v is the unique bilinear differential operator (i.e. a bilinear combination of the derivatives of f and g), up to rescaling, which gives a modular form of weight $k + \ell + 2v$. Furthermore, for $v > 0$ the resulting modular forms are cusp forms, which hold special significance in the literature.

The collection of Rankin–Cohen brackets (over all k, ℓ and v) endow rich algebraic structure on the graded algebra of modular forms as described by Zagier [Zag94]. Their algebraic properties have been the subject of further study, for instance by El Gradechi [EG06]. As well as being interesting in their own right, Rankin–Cohen operators are more generally used as tools in the theory of modular forms. I was first introduced to Rankin–Cohen operators following their application in [Eis+21] in the setting of mod p modular forms, but they have also been used in the study of L-values and of Galois representations, for example. This diverse range of applications provides further motivation for their study.

2.2. **Generalizations to Automorphic Forms.** My recent research builds on several previous works generalizing Rankin–Cohen-type operators beyond modular forms for $\mathrm{SL}_2(\mathbb{Z})$. Namely, the goal is to construct and study analogous operators with similar properties as those defined above in different settings. The particular case I have worked with is that of *Hermitian modular forms*, which are holomorphic functions (again valued in $\mathbb{C}$) on an "upper half-space" of n-by-n matrices

$$\mathbb{H}_n := \left\{ Z \in M_{n,n}(\mathbb{C}) \mid \frac{1}{2i}(Z - Z^*) \text{ is positive definite} \right\},$$

where Z^* denotes the conjugate-transpose of Z. Generalizing the action of $\mathrm{SL}_2(\mathbb{Z})$ by Möbius transforms, we now consider the *unitary group of signature* (n, n)

$$U(n,n) := \left\{ \gamma \in M_{2n,2n}(\mathcal{O}_L) \mid \gamma^* \begin{pmatrix} 0 & -i1_n \\ i1_n & 0 \end{pmatrix} \gamma = \begin{pmatrix} 0 & -i1_n \\ i1_n & 0 \end{pmatrix} \right\},$$

with coefficients in the ring of integers $\mathcal{O}_L$ of a fixed quadratic imaginary extension L of a totally real number field L^+. This group acts on $\mathbb{H}_n$ by

$$\gamma \cdot Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z := (AZ + B)(CZ + D)^{-1}$$

for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(n,n)$ and $Z \in \mathbb{H}_n$. A Hermitian modular form F then needs to satisfy

$$F(Z) = \det(CZ + D)^{-k} F(\gamma \cdot Z),$$

where again k is a fixed non-negative integer, the weight of F.

Pre-dating my work, Choie–Eholzer [CE98] used Jacobi forms to construct Rankin–Cohen type operators for Siegel modular forms, a different higher-dimensional generalization of modular forms, which Eholzer–Ibukiyama [EI98] and Ibukiyama [Ibu99] further extended using a different approach involving theory of pluriharmonic polynomials, in part developed by Kashiwara–Vergne [KV78]. In the case of Hermitian modular forms, Martin–Senadheera [MS17] used Jacobi forms to produce analogues of Rankin–Cohen brackets for Hermitian modular forms over the field extension $\mathbb{Q}(i)/\mathbb{Q}$ and signature $(2, 2)$.

I employed a similar approach to [EI98], reframing the problem using a correspondence between differential operators and polynomials, and again using results of [KV78]. Without further restriction on the underlying field L/L^+ and in signature (n, n), I proved the following results:

Theorem (A)**.** (Imprecise version of Theorem 11 and Corollary 13, [Dun24].) Covariant multilinear differential operators sending r scalar-valued Hermitian modular forms of integer weights $k_1, \ldots, k_r$ respectively to a scalar-valued Hermitian modular form of weight $v + \sum_{i=1}^{r} k_i$ for a fixed $v \geq 0$ correspond to a certain space of homogeneous pluri-harmonic polynomials.

Theorem (B)**.** (Imprecise version of Theorem 22 and Proposition 25, [Dun24].) In the case $r = 2$, for each $v \geq 0$ and $n > 1$ there is a unique (up to rescaling) bilinear differential operator sending scalar-valued Hermitian modular forms of weights k_1 and k_2 respectively to a Hermitian modular form of weight $k_1 + k_2 + 2v$. I also give integral linear relations satisfied by the coefficients of the differential operators, which implies that a normalization can be chosen to preserve the ring containing the Fourier coefficients. Furthermore, for $v > 0$ the resulting Hermitian modular form is always a cusp form.

Namely, I produced Rankin–Cohen type operators on Hermitian modular forms in a more general setting than before, and showed that these operators satisfy some important properties analogous to those of the original Rankin–Cohen operators highlighted above.

3. Future Work

There are several directions in which I am either currently working or would like to pursue in the future. Immediately building off my previous work, a slight generalization to my results in [Dun24] is currently in progress, to cover unitary groups of different signatures, i.e. automorphic forms on $U(m, n)$ with $m \neq n$. Furthermore, I would like to obtain a closed formula for the operators constructed *loc cit.*. One could also consider a generalization to vector-valued Hermitian modular forms of more general weight, although this may be less straightforward.

A project with much wider scope is an exploration of the Lie-theoretic properties of generalized Rankin–Cohen operators. For instance, a greater understanding of this aspect might be gained by a detailed comparison of my construction to that of Ban [Ban06] who worked in a similar (but not identical) setting but from the perspective of representation theory.

One could go on to consider questions such as the decomposition of cuspidal automorphic representations induced by these operators.

Additionally, it would be interesting to consider the interplay with other types of differential operators. For example, in the elliptic setting Lanphier [Lan08] proved a relationship between the Maass–Shimura operators and the Rankin–Cohen brackets. One can then formulate a geometric analogue of these differential operators in terms of the Gauss–Manin connection. So, following a similar line of inquiry may yield a better understanding of the geometry in this setting.

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