Learning and Expectations in Macroeconomics

Problems for Chapter 11

1. Consider the Overlapping Generations model with a multiplicative productivity shock, i.e. output is

$$Q_t = n_t v_t \text{ where } v_t \overset{iid}{\sim} U[1-\alpha, 1+\alpha], \; 0 < \alpha < 1.$$

Here $U[1-\alpha, 1+\alpha]$ denotes the uniform distribution over $[1-\alpha, 1+\alpha]$. Agents maximize

$$E_t\left(U(c_{t+1}) - V(n_t)\right) \text{ subject to } p_t n_t v_t = p_{t+1} c_{t+1}.$$

Let

$$U(c) = c^{1-\sigma}/(1-\sigma) \text{ and } V(n) = n^{1+\varepsilon}/(1+\varepsilon) \text{ where } \sigma, \varepsilon > 0.$$

(a) Show that with constant money stock the equilibrium equation satisfies

$$n_t = \left(E_t(Q_{t+1}^{1-\sigma})\right)^{1/(1+\varepsilon)}.$$

(b) Under adaptive learning we assume that

$$\begin{aligned} n_t &= (X_{t+1}^e)^{1/(1+\varepsilon)} \\ Q_t &= n_t v_t, \end{aligned}$$

where X_{t+1}^e is the agents' forecast of $q_{t+1}^{1-\sigma}$, given by

$$X_{t+1}^e = X_t^e + t^{-1}\left(Q_{t-1}^{1-\sigma} - X_t^e\right).$$

Using Matlab or another programimg language, simulate the system for $T = 10,000$ periods. Set $\sigma = 0.5$, $\varepsilon = 0.1$ and $\alpha = 0.5$. Choose initial $X_1^e = 0.5$.

(i) Plot X_t^e and Q_t over the first 150 periods.

(ii) Calculate the sample mean and standard deviation of q_t and X_t^e over the last 1000 periods.

2. Suppose

$$y_t = E_t F(y_{t+1}) + v_t,$$

where v_t is *iid* with bounded support and mean 0. Restrict attention to solutions with $y_t > 0$ and suppose that F is continuous with a unique positive fixed point $\bar{y} = F(\bar{y})$. Assume also that (i) F is concave and (ii) $F(y) - y > 0$ for $y < \bar{y}$ and $F(y) - y < 0$ for $y > \bar{y}$. Suppose that there exists a noisy steady state of the form

$$y_t = \bar{\theta} + v_t.$$

Using Jensen's inequality, show that $\bar{\theta} \leq \bar{y}$.

3. Consider the OG model with additive productivity shocks (Example 2 of Section 11.3.1), with the the utility specification given in Example 3, i.e. $U(c) = c^{1-\sigma}/(1-\sigma)$ and $V(n) = n^{1+\varepsilon}/(1+\varepsilon)$ where $\sigma, \varepsilon > 0$, and where $\lambda_t = \beta + v_t$ with v_t *iid* uniform over $[-\alpha, \alpha]$.

   (a) Show that

$$(n_t + \lambda_t)n_t^{\varepsilon} = E_t^*(n_{t+1} + \lambda_{t+1})^{1-\sigma}.$$

   (b) Show that when $\varepsilon = 1$ the model can be written in the form $n_t = H(E_t^* G(y_{t+1}, v_{t+1}), v_t)$, where

$$\begin{aligned} G(n, v) &= (n + \beta + v)^{1-\sigma} \\ H(\theta, v) &= 0.5\left(-\beta - v + ((\beta + v)^2 + 4\theta)^{0.5}\right). \end{aligned}$$

   (c) Recall that a noisy steady state satisfies $y_t = H(\bar{\theta}, v_t)$ where $\bar{\theta}$ satisfies

$$\bar{\theta} = T(\bar{\theta}) \text{ where } T(\theta) \equiv EG(H(\theta, v), v).$$

   Write a program, using for example Matlab, that numerically computes the noisy steady state $\bar{\theta}$. Assuming $\sigma = 4.0$, find the values of $\bar{\theta}$ for $\alpha = 0.001, \alpha = 0.1$ and $\alpha = 0.3$. (This can be done using a numerical integration comand to compute the required expectation and a grid search to find the equilibrium value $\bar{\theta}$).

   (d) For each value of α in (c), compute the derivative $T'(\bar{\theta})$ numerically. In which cases is the RE steady state stable under the adaptive learning rule given in Section 11.4?

   (e) Anticipating Chapter 12, we say that a nosiy steady state $\bar{\theta}$ is strongly stable under adaptive learning if it remains locally stable even when agents overparameterize it as a 2-cycle. The corresponding strong E-stability condition is $\left|T'(\bar{\theta})\right| < 1$. For each value of α in (c) determine whether $\bar{\theta}$ is strongly E-stable.

4. Consider the hyperinflation model with linear savings function, so that inflation in period t is given by

$$\pi_t = \frac{a - b\pi_t^e}{a - b\pi_{t+1}^e - g}.$$

   The agents are assumed to use the learning rule with constant gain

$$\pi_{t+1}^e = \pi_t^e + \gamma(\pi_{t-1} - \pi_t^e).$$

   (a) Introducing the notation $x_t = \pi_{t-1}^e$, show that the dynamics can be written in the form

$$\begin{aligned} \Delta\pi_{t+1}^e &\equiv \pi_{t+1}^e - \pi_t^e = f(\pi_t^e, x_t) \\ \Delta x_{t+1} &= x_{t+1} - x_t = g(\pi_t^e, x_t). \end{aligned}$$

   Derive the forms of the two functions. Note that we have the restrictions $0 \leq \pi_t^e, x_t \leq a/b$.

(b) Construct a phase diagram in the (x_t, π_t^e) -space by drawing the loci $\Delta\pi_{t+1}^e = 0$ and $\Delta x_{t+1} = 0$. Illustrate the dynamics in the usual way and show that there are divergent paths in this dynamics.
(c) The model has usually two steady states. Derive the condition for local stability of the learning dynamics.
(Note: since these are difference equations, the phase diagram does not fully describe the dynamic possibilities since there could be jumps over the equilibrium loci.)

5. In the monetary inflation model it is assumed that nominal money growth is constant $M_t = \theta M_{t-1}$ and that government expenditure is determined endogenously by from the seignorage equation.
(a) Denote the savings function as $M_t/P_t = S(\pi_{t+1}^e)$. Show that the temporary equilibrium is given by

$$\pi_t = \frac{\theta S(\pi_t^e)}{S(\pi_{t+1}^e)}.$$

Determine the possible steady states of this model.
(b) Derive the condition for E-stability of the steady state, when the PLM of the agents takes the form $\pi_t^e = a$, constant.
(c) Assume that agents have the learning rule

$$\pi_{t+1}^e = \pi_t^e + \gamma_{t+1}(\pi_{t-1} - \pi_t^e).$$

Study it under both decreasing gain and constant gain $(\gamma_{t+1} = \gamma)$ and determine the convergence condition for a steady state.