Learning and Expectations in Macroeconomics
Problems for Chapter 2

1. Consider the standard least squares formula

$$c = (\sum_{i=1}^{T} x_i x_i')^{-1} (\sum_{i=1}^{T} x_i y_i).$$

This formula arises in fitting the regression equation $y_i = c' x_i + e_i$ using data $i = 1, ..., T$ on the $k \times 1$ independent vector x_i and the dependent variable y_i, so that c minimizes $\sum_{i=1}^{T} e_i^2$. Writing

$$\begin{aligned} R_t &= t^{-1} \sum_{i=1}^{t} x_i x_i' \\ c_t &= t^{-1} R_t^{-1} \sum_{i=1}^{t} x_i y_i \end{aligned}$$

for the moment matrix and the coefficient vector show by mathematical induction on the number of data points that c can instead be computed using the *recursive least squares (RLS)* formulae

$$\begin{aligned} c_t &= c_{t-1} + t^{-1} R_t^{-1} x_t (y_t - x_t' c_{t-1}) \\ R_t &= R_{t-1} + t^{-1} (x_t x_t' - R_{t-1}). \end{aligned}$$

2. Consider the model

$$p_t = f(p_{t+1}^e).$$

Suppose that $p_{t+1}^e = a_t$, where $a_t = a_{t-1} + \gamma_t (p_{t-1} - a_{t-1})$, so that

$$a_t = a_{t-1} + \gamma_t (f(a_{t-1}) - a_{t-1}).$$

(a) For the decreasing gain case $\gamma_t \to 0$, apply the stochastic approximation technique to obtain the condition for stability under learning of a steady state $\bar{a} = f(\bar{a})$. That is, treating the above equation as an SRA, show how to obtain the assocaited ODE and find the correspoding stability condition.

(b) For the constant gain case $\gamma_t = \gamma$ for $0 < \gamma \leq 1$, derive the local stability condition for a steady state $\bar{a} = f(\bar{a})$ using the standard results on the local stability of difference equations.

3. Consider a variation of the cobweb model in which p_t depends on the observable exogenous variable w_t rather than w_{t-1}:

$$\begin{aligned} p_t &= \mu + \alpha p_t^e + \delta w_t + \eta_t, \text{ where } \alpha \neq 1, \\ w_t &= k + \lambda w_{t-1} + \varepsilon_t, \text{ where } |\lambda| < 1, \end{aligned}$$

where ε_t and η_t are independent white noise processes. (Here w_t is univariate and η_t is unobserved).

(a) Obtain the unique REE and show that it can be written in the form

$$p_t = \bar{a} + \bar{b}w_{t-1} + u_t,$$

for suitable $\bar{a}, \bar{b}$ and u_t white noise.

(b) Suppose agents forecast according to

$$p_t^e = a_{t-1} + b_{t-1}w_{t-1},$$

where a_{t-1}, b_{t-1} are estimated in the usual way by RLS ("least squares learning").

(i) For the PLM $p_t = a + bw_{t-1} + u_t$, obtain the T-mapping from the PLM to the ALM and find the E-stability condition for the REE

(ii) Outline the steps of the stochastic approximation argument used to obtain the stability condition for the REE under least squares learning and show that the condition is identical to the E-stability condition.

4. Consider the Cagan model

$$\begin{aligned} y_t &= \mu + \beta E_t^* y_{t+1} + \delta w_t \\ w_t &= \lambda w_{t-1} + v_t. \end{aligned}$$

The parameter λ is assumed to be known.

(a) Show that generically there exists a unique REE of the form

$$y_t = \bar{a} + \bar{b}w_t.$$

(b) Suppose agents have a PLM of the form $y_t = a + bw_t$ and forecast accordingly. Derive the $T-$mapping and the E-stability conditions.

5. Continuing on Problem 4, suppose agents update the parameters of their PLM

$$y_t = a_{t-1} + b_{t-1}w_t$$

using the RLS algorithm. Write down the details of the algorithm. Using numerical parameter values $\mu = 5$, $\beta = 0.8$ and $\delta = 1$, write a Matlab routine to simulate the RLS learning and show the results for a_t, b_t and y_t.